Robustness analysis techniques
for clearance of flight control laws

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COFCLUO project

Clearance Of Flight Control Laws Using Optimization
Funded by EC, 2007-2010

Partners:
- Linköping University (Svezia)
- AIRBUS (Francia)
- DLR (Germania)
- ONERA (Francia)
- FOI (Svezia)
- Università di Siena (Italy)
**COFCLUO project**

Motivations:

- Validation of FCLs is a key issue in terms of time and costs
- Baseline solution in industry mainly relies on brute force simulation in a huge number of flight points

Objectives:

- Detect worst-cases faster and/or more reliably than using Monte Carlo based techniques
- Give guarantees for whole regions of flight envelope to be cleared
Models

Two types of models used within COFCLUO project:

- Complex Simulink models of closed-loop aircraft dynamics used for worst-case detection (provided by AIRBUS)
- Linear Fractional Representation (LFR) models, including rigid and flexible modes, for clearing whole regions

LFR models derived from AIRBUS physical models (done by DLR and ONERA)
UNISI contribution

Techniques for clearing entire regions of the flight envelope:

- robust aeroelastic stability (this talk)
- comfort criterion (this talk)
- robust stability of systems with saturations (Talk FrB06.1 in Milan)
Robust aeroelastic stability

The largest real part of the closed-loop eigenvalues has to be negative, for all possible values taken by the uncertain parameters (aircraft mass configuration) and the trimmed flight variables (Mach number and calibrated air speed).

Techniques adopted:
- Lyapunov-based analysis (UNISI)
- $\mu$ analysis (ONERA)
- IQCs (Linköping)
Robust stability of LFR uncertainty models

Consider the LFR system

\[
\Sigma : \quad \dot{x}(t) = A(\theta)x(t) = \left[ A + B\Delta(\theta)(I - D\Delta(\theta))^{-1}C \right] x(t)
\]

where \( \Delta(\theta) = \text{diag}(\theta_1 I_{s_1}, \ldots, \theta_{n_\theta} I_{s_{n_\theta}}) \)

An equivalent representation of \( \Sigma \) is given by:

\[
\Sigma : \quad \begin{cases} 
\dot{x} = Ax + Bq \\
\quad p = Cx + Dq \\
\quad q = \Delta(\theta)p
\end{cases}
\]

where \( x \in \mathbb{R}^n, q, p \in \mathbb{R}^d \) and \( d = \sum_{i=1}^{n_\theta} s_i \).

Assumptions:

\[ \theta \in \Theta = [\underline{\theta}_1, \overline{\theta}_1] \times \cdots \times [\underline{\theta}_{n_\theta}, \overline{\theta}_{n_\theta}] \quad \text{with} \quad 2^{n_\theta} \quad \text{vertices} \quad \text{Ver}[\Theta] \]

\[ \dot{\theta}(t) = 0 \quad \text{(time-invariant uncertainty)} \]
Three methods based on parameter-dependent LFs

Dettori & Scherer (2000)
Fu & Dasgupta (2001)

Methods based on polynomial LFs not suitable due to models size
[Dettori & Scherer '00]

If there exist:

- a symmetric Lyapunov matrix $P(\theta) \in \mathbb{R}^{n \times n}$, multiaffine in $\theta$
- two matrices $S_0, S_1 \in \mathbb{R}^{d \times d}$

such that $\forall \theta \in \text{Ver}[\Theta]$

$$P(\theta) > 0 \quad \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & P(\theta) & 0 \\ P(\theta) & 0 & 0 \\ 0 & 0 & W(\theta) \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix} < 0,$$

where

$$W(\theta) = \begin{bmatrix} S_1 + S_1^T & -S_0 - S_1 \Delta(\theta) \\ -S_0^T - \Delta(\theta) S_1^T & S_0^T \Delta(\theta) + \Delta(\theta) S_0 \end{bmatrix},$$

then the system $\Sigma$ is robustly stable.
Remarks:

- multiaffine LF

\[ V(x; \theta) = x^T \left( P_0 + \sum_{j=1}^{n_\theta} \theta_j P_j + \sum_{i=1}^{n_\theta} \sum_{j=i+1}^{n_\theta} \theta_i \theta_j P_{ij} + \cdots \right) x \]

- parameter-dependent multiplier \( W(\theta) \), parameterized by \( S_0, S_1 \)
- \( 2^{n_\theta} \) LMIs of dimension \((n + d)\), \(2^{n_\theta}\) LMIs of dimension \(n\)
- \( 2d^2 + 2^{n_\theta} \frac{n(n+1)}{2} \) free variables
[Fu & Dasgupta '01]

Let $T_i = \text{blockdiag}(0_{s_1}, \ldots, 0_{s_{i-1}}, I_{s_i}, 0_{s_{i+1}}, \ldots, 0_{s_{n_\theta}})$, $C_i = T_i C$, $D_i = T_i D$ for $i = 1, \ldots, n_\theta$, and $D_0 = -I$.

If there exist $2n_\theta + 2$ matrices $C_{\mu,i} \in \mathbb{R}^{d \times n_\theta}$, $D_{\mu,i} \in \mathbb{R}^{d \times d}$, $i = 0, \ldots, n_\theta$, s.t.

$$\begin{bmatrix} C_i^T & D_{\mu,i}^T \\ D_i & C_{\mu,i} \end{bmatrix} \begin{bmatrix} C_i & D_i \\ C_{\mu,i} & D_{\mu,i} \end{bmatrix} \leq 0, \quad i = 1, \ldots, n_\theta$$

and a symmetric $P(\theta) \in \mathbb{R}^{n \times n}$, multiaffine in $\theta$, such that $\forall \theta \in \text{Ver}[\Theta]$

$$P(\theta) > 0$$

$$\begin{bmatrix} A^T(\theta)P(\theta) + P(\theta)A(\theta) & \Pi(\theta) \\ \Pi^T(\theta) & -(D_{\mu}(\theta)D^{-1}(\theta) + D^{-T}(\theta)D_{\mu}^T(\theta)) \end{bmatrix} < 0,$$

where $\Pi(\theta) = (P(\theta)BD^{-1}(\theta) - C_{\mu}(\theta) + C(\theta)D^{-T}(\theta)D_{\mu}^T(\theta))$

$C(\theta) = \Delta(\theta)C$, $D(\theta) = \Delta(\theta)D - I$,

$C_{\mu}(\theta) = C_{\mu,0} + \sum_{i=1}^{n_\theta} \theta_i C_{\mu,i}$, $D_{\mu}(\theta) = D_{\mu,0} + \sum_{i=1}^{n_\theta} \theta_i D_{\mu,i}$

then the system $\Sigma$ is robustly stable.
Remarks:

- multiaffine LF

- parameter-dependent multipliers \(C_\mu(\theta)\) and \(D_\mu(\theta)\), parameterized by \(C_{\mu,i}, D_{\mu,i}, i = 0, \ldots, n_\theta\)

- \(n_\theta + 2^{n_\theta}\) LMIs of dimension \((n + d)\), \(2^{n_\theta}\) LMIs of dimension \(n\)

- \((n_\theta + 1)(nd + d^2) + 2^{n_\theta} \frac{n(n + 1)}{2}\) free variables
[Wang & Balakrishnan '02]

If there exist:

\[ n_\theta + 1 \] symmetric matrices \( Q_0, \ldots, Q_{n_\theta} \in \mathbb{R}^{n \times n} \)

a symmetric scaling matrix \( N \in \mathbb{R}^{d \times d} \)

such that, \( \forall \theta \in \text{Ver}[\Theta], \)

\[
\begin{cases}
N > 0 \\
Q(\theta) = Q_0 + \sum_{j=1}^{n_\theta} \theta_j Q_j > 0 \\
\begin{bmatrix}
AQ(\theta) + Q(\theta)A^T + B\Delta(\theta)N\Delta(\theta)B^T \\
(Q(\theta)C^T + B\Delta(\theta)N\Delta(\theta)D^T)^T
\end{bmatrix} < 0
\end{cases}
\]

then the system \( \Sigma \) is robustly stable.
Remarks:

- candidate LF

\[ V(x) = x^T \left( Q_0 + \sum_{j=1}^{n_\theta} \theta_j Q_j \right)^{-1} x \]

- \(2^{n_\theta}\) LMIs of dimension \((n + d)\), \(2^{n_\theta}\) LMIs of dimension \(n\), 1 LMI of dimension \(d\)

- \(\frac{d(d+1)}{2} + (n_\theta + 1) \frac{n(n+1)}{2}\) free variables

- easily extended to slowly time-varying parameters

- generalized to polynomial Lyapunov functions [Chesi et al., '04]
Dealing with complexity

The considered methods are generally computationally unfeasible for the clearance problems at hand

⇒ Find appropriate *relaxations*
   (trade off conservatism and computational burden)

⇒ Divide into simpler problems
   (partition the uncertainty domain)

Strong requirement: easy-to-use tools
Relaxations

- Lyapunov function: multiaffine (mapdlf), affine (apdlf), constant (clf)
- Multipliers: affine, constant, diagonal
- Scalings: constant, diagonal

<table>
<thead>
<tr>
<th>Relaxation</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD-$c_\mu$</td>
<td>FD method with constant multipliers $C_{\mu,0}$, $D_{\mu,0}$</td>
</tr>
<tr>
<td>FD-$cd_\mu$</td>
<td>FD method with constant diagonal multipliers $C_{\mu,0}$, $D_{\mu,0}$</td>
</tr>
<tr>
<td>DS-$dS$</td>
<td>DS method with diagonal multipliers $S_0$, $S_1$</td>
</tr>
<tr>
<td>WB-$dN$</td>
<td>WB method with diagonal scaling matrix $N$</td>
</tr>
</tbody>
</table>
Progressive tiling

LTI uncertainty: partition the uncertainty domain into rectangular tiles, then test robustness in each tile

Algorithm:

1) start with an hyperbox containing the entire uncertainty domain $\Theta$
   if cleared then: done! if not
2) reduce the size of the uncleared tiles (by bisecting each side)
3) repeat until every tile is cleared or minimum tile size is reached

Remark: for each tile the LFR is re-parameterized.
Adaptive tiling

Idea: combine progressive tiling approach with an adaptive choice of the relaxation (different Lyapunov function or multiplier)

Rationale: use conservative but fast methods first, then switch to more powerful and computationally demanding ones only for the uncleared tiles

Tested on clearance problems: adaptation on the Lyapunov function constant $\rightarrow$ affine $\rightarrow$ multiaffine
**Gridding**

Before attempting to clear a tile, the tile is gridded and stability of models on the grid is checked.

If at least one model on the grid is unstable, the tile is skipped and temporarily marked as unstable. (portions of the tile can be later cleared, as partitioning proceeds)

Three types of tiles when max number of partitions is reached:

- **Cleared**
- **Unstable** (contain unstable models found by gridding)
- **Unknown** (not cleared and no unstable models)
Software and GUI

All techniques implemented in MATLAB using:

- LFR toolbox
- YALMIP
- SDPT3

A Graphical User Interface (GUI) developed to set up clearance problems and display results
Example of GUI output

Green tiles: cleared
Red tiles: unstable (contain unstable models found by gridding)
White tiles: unknown (not cleared and no unstable models)
**Main GUI features**

**Inputs:**
- model selection
- choice of methods, relaxations and tiling options
- restriction to polytopic flight envelopes
- shifted stability (for slowly divergent modes)

**Outputs:**
- 2D plots of cleared, unstable and unknown regions
- number of optimization problems solved and elapsed time
- rates of cleared, unstable and unknown domain

*clearance rate*: ratio between cleared region and “clearable” domain (tiles that do not contain unstable models found by gridding).
New version available

A new version of the GUI is available at:

www.dii.unisi.it/~garulli/lfr_rai/

... try it! (any feedback is welcome)
Closed-loop integral longitudinal models

Uncertain parameters and trim flight variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Nominal value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>central tank</td>
<td>0.5</td>
</tr>
<tr>
<td>$O$</td>
<td>outer tank</td>
<td>0</td>
</tr>
<tr>
<td>$P$</td>
<td>payload</td>
<td>0.5</td>
</tr>
<tr>
<td>$X$</td>
<td>center of gravity</td>
<td>0</td>
</tr>
<tr>
<td>$M$</td>
<td>Mach number</td>
<td>0.86</td>
</tr>
<tr>
<td>$V$</td>
<td>calibrated air speed</td>
<td>310 kt</td>
</tr>
</tbody>
</table>
**LFR models for aeroelastic stability**

Models representative of closed-loop aeroelastic longitudinal dynamics in frequency range [0,15] rad/sec.

<table>
<thead>
<tr>
<th>Model</th>
<th>$n$</th>
<th>$d$</th>
<th>$\theta_1, s_1$</th>
<th>$\theta_2, s_2$</th>
<th>$\theta_3, s_3$</th>
<th>$\theta_4, s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>20</td>
<td>16</td>
<td>$C, 16$</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>CX</td>
<td>20</td>
<td>18</td>
<td>$C, 14$</td>
<td>–</td>
<td>–</td>
<td>$X, 4$</td>
</tr>
<tr>
<td>OC</td>
<td>20</td>
<td>50</td>
<td>$C, 26$</td>
<td>$O, 24$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>OCX</td>
<td>20</td>
<td>50</td>
<td>$C, 24$</td>
<td>$O, 22$</td>
<td>–</td>
<td>$X, 4$</td>
</tr>
<tr>
<td>POC</td>
<td>20</td>
<td>79</td>
<td>$C, 42$</td>
<td>$O, 24$</td>
<td>$P, 13$</td>
<td>–</td>
</tr>
<tr>
<td>MV</td>
<td>20</td>
<td>54</td>
<td>$M, 26$</td>
<td>$V, 28$</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Parameters not appearing in $\Delta$ block are set to the nominal values.
Uncertainty set

MV model

Flight variables $M$ and $V$ are bounded in a polytope, representing the considered flight envelope.

Robustness analysis has been carried out on the smallest rectangle including the polytope.

Other models

Fuel loads ($C$ and $O$), payload ($P$) and position of center of gravity ($X$) take normalized values between 0 and 1.

Corresponding uncertainty domains are hyper-boxes in the appropriate dimensions.
**Aeroelastic stability results: C and CX models**

**C model: progressive tiling**

<table>
<thead>
<tr>
<th>Method (lf)</th>
<th>Cleared</th>
<th>NOPs</th>
<th>t (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS (clf)</td>
<td>1</td>
<td>1</td>
<td>9.78</td>
</tr>
<tr>
<td>DS (apdlf)</td>
<td>1</td>
<td>1</td>
<td>10.97</td>
</tr>
<tr>
<td>DS-dS (clf)</td>
<td>1</td>
<td>3</td>
<td>10.15</td>
</tr>
<tr>
<td>DS-dS (apdlf)</td>
<td>1</td>
<td>1</td>
<td>4.15</td>
</tr>
<tr>
<td>FD-c$\mu$ (clf)</td>
<td>1</td>
<td>1</td>
<td>5.81</td>
</tr>
<tr>
<td>FD-c$\mu$ (apdlf)</td>
<td>1</td>
<td>1</td>
<td>8.77</td>
</tr>
<tr>
<td>FD-cd$\mu$ (clf)</td>
<td>1</td>
<td>3</td>
<td>8.43</td>
</tr>
<tr>
<td>FD-cd$\mu$ (apdlf)</td>
<td>1</td>
<td>1</td>
<td>4.93</td>
</tr>
<tr>
<td>WBQ-dM</td>
<td>1</td>
<td>3</td>
<td>4.38</td>
</tr>
</tbody>
</table>

**CX model: progressive tiling**

<table>
<thead>
<tr>
<th>Method (lf)</th>
<th>Cleared</th>
<th>NOPs</th>
<th>t (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS (clf)</td>
<td>1</td>
<td>1</td>
<td>30.88</td>
</tr>
<tr>
<td>DS (apdlf)</td>
<td>1</td>
<td>1</td>
<td>44.47</td>
</tr>
<tr>
<td>DS-dS (clf)</td>
<td>1</td>
<td>5</td>
<td>48.59</td>
</tr>
<tr>
<td>DS-dS (apdlf)</td>
<td>1</td>
<td>1</td>
<td>16.33</td>
</tr>
<tr>
<td>FD-c$\mu$ (clf)</td>
<td>1</td>
<td>1</td>
<td>18.60</td>
</tr>
<tr>
<td>FD-c$\mu$ (apdlf)</td>
<td>1</td>
<td>1</td>
<td>32.30</td>
</tr>
<tr>
<td>FD-cd$\mu$ (clf)</td>
<td>1</td>
<td>5</td>
<td>32.37</td>
</tr>
<tr>
<td>FD-cd$\mu$ (apdlf)</td>
<td>1</td>
<td>1</td>
<td>18.91</td>
</tr>
<tr>
<td>WBQ-dM</td>
<td>1</td>
<td>5</td>
<td>9.16</td>
</tr>
</tbody>
</table>

Cleared: rate of uncertainty domain cleared  
NOPs: number of LMI tests performed
Aeroelastic stability results: **OC and OCX models**

**OC model: progressive tiling**

<table>
<thead>
<tr>
<th>Method (lf)</th>
<th>Cleared</th>
<th>NOPs</th>
<th>t (h:m:s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS-dS (clf)</td>
<td>1</td>
<td>73</td>
<td>0:26:32</td>
</tr>
<tr>
<td>DS-dS (apdlf)</td>
<td>1</td>
<td>41</td>
<td>0:31:51</td>
</tr>
<tr>
<td>FD-(c\mu) (clf)</td>
<td>1</td>
<td>33</td>
<td>2:55:30</td>
</tr>
<tr>
<td>FD-(c\mu) (apdlf)</td>
<td>1</td>
<td>1</td>
<td>0:06:11</td>
</tr>
<tr>
<td>FD-(cd\mu) (clf)</td>
<td>1</td>
<td>85</td>
<td>0:22:00</td>
</tr>
<tr>
<td>FD-(cd\mu) (apdlf)</td>
<td>1</td>
<td>49</td>
<td>0:31:50</td>
</tr>
<tr>
<td>WBQ-dM</td>
<td>1</td>
<td>169</td>
<td>0:14:48</td>
</tr>
</tbody>
</table>

**OCX model: progressive tiling**

<table>
<thead>
<tr>
<th>Method (lf)</th>
<th>Cleared</th>
<th>NOPs</th>
<th>t (h:m:s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS (clf)</td>
<td>1</td>
<td>185</td>
<td>327:00:28</td>
</tr>
<tr>
<td>DS (apdlf)</td>
<td>1</td>
<td>1</td>
<td>3:07:22</td>
</tr>
<tr>
<td>DS-dS (clf)</td>
<td>1</td>
<td>745</td>
<td>11:10:09</td>
</tr>
<tr>
<td>DS-dS (apdlf)</td>
<td>1</td>
<td>265</td>
<td>10:53:36</td>
</tr>
<tr>
<td>FD-(c\mu) (clf)</td>
<td>1</td>
<td>185</td>
<td>41:31:34</td>
</tr>
<tr>
<td>FD-(c\mu) (apdlf)</td>
<td>1</td>
<td>1</td>
<td>0:17:47</td>
</tr>
<tr>
<td>FD-(cd\mu) (clf)</td>
<td>1</td>
<td>841</td>
<td>8:52:25</td>
</tr>
<tr>
<td>FD-(cd\mu) (apdlf)</td>
<td>1</td>
<td>385</td>
<td>13:34:34</td>
</tr>
<tr>
<td>WBQ-dM</td>
<td>1</td>
<td>2129</td>
<td>4:34:12</td>
</tr>
</tbody>
</table>
**Aeroelastic stability results: POC model**

POC model: progressive tiling

<table>
<thead>
<tr>
<th>Method</th>
<th>Cleared</th>
<th>OPS</th>
<th>t (h:m:s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS-dS (clf)</td>
<td>1</td>
<td>993</td>
<td>33:42:48</td>
</tr>
<tr>
<td>FD-c_{\mu} (clf)</td>
<td>1</td>
<td>105</td>
<td>142:38:10</td>
</tr>
</tbody>
</table>
## Aeroelastic stability results: MV model

**MV model: progressive tiling**

<table>
<thead>
<tr>
<th>Method</th>
<th>Rate</th>
<th>NOPs</th>
<th>Time (h:m:s)</th>
<th>Time/OP (h:m:s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS-dS (apdlf)</td>
<td>0.9931</td>
<td>1030</td>
<td>24 : 35 : 25</td>
<td>0 : 01 : 25</td>
</tr>
<tr>
<td>DS-dS (clf)</td>
<td>0.9875</td>
<td>1282</td>
<td>12 : 32 : 13</td>
<td>0 : 00 : 35</td>
</tr>
<tr>
<td>FD-cµ (apdlf)</td>
<td>1</td>
<td>174</td>
<td>30 : 09 : 45</td>
<td>0 : 10 : 24</td>
</tr>
<tr>
<td>FD-cµ (clf)</td>
<td>0.9993</td>
<td>218</td>
<td>34 : 00 : 29</td>
<td>0 : 09 : 21</td>
</tr>
<tr>
<td>FD-cdµ (apdlf)</td>
<td>0.9921</td>
<td>1202</td>
<td>20 : 59 : 54</td>
<td>0 : 01 : 02</td>
</tr>
<tr>
<td>FD-cdµ (clf)</td>
<td>0.9895</td>
<td>1346</td>
<td>8 : 28 : 02</td>
<td>0 : 00 : 22</td>
</tr>
</tbody>
</table>

Rate: clearance rate (cleared / clearable)
DS-ds with clf

DS-ds with apdlf
FD-$c_\mu$ with clf

FD-$c_\mu$ with apdlf
FD-$\text{cd}_\mu$ with clf

FD-$\text{cd}_\mu$ with apdlf
**Adaptive tiling**

**MV model: adaptive tiling**

<table>
<thead>
<tr>
<th>Method</th>
<th>clf</th>
<th>apdlf</th>
<th>Rate</th>
<th>NOPs</th>
<th>Time (h:m:s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DS-dS</td>
<td>0</td>
<td>7</td>
<td>0.9931</td>
<td>1030</td>
<td>24:35:25</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5</td>
<td>0.9931</td>
<td>1042</td>
<td>27:27:54</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>0.9931</td>
<td>1110</td>
<td>25:20:00</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>1</td>
<td>0.9931</td>
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Comments

apdlf conditions need to solve fewer optimization problems than clf ones but this reduces the computational time only if number of problems is significantly smaller (compare FD-\(c_\mu\) and FD-\(cd_\mu\) for OC and OCX)

choosing structurally simpler multipliers can increase the time if the number of optimization problems grows too much (see FD with apdlf for OC and OCX)

choosing the most powerful method is not always wise (or possible)
difficult to pick “best” method \textit{a priori}

adaptation of LF structure has reduced times for DS-ds (15%) and FD-\(cd_\mu\) (25%), not for FD-\(c_\mu\) (again: difficult to predict this \textit{a priori})
Comfort analysis

Joint work with Anders Hansson and Ragnar Wallin (Linköping University)

Comfort index formulated as $\mathcal{H}_2$ performance from wind velocity to acceleration at specific points along the aircraft fuselage

Need robust $\mathcal{H}_2$ analysis to account for uncertainty in the flight parameters

Limited frequency range:
- industrial practice computes comfort index on a limited freq interval
- LFR models are valid only in specific freq range
**Comfort index**

\[ J_c = \sqrt{\frac{1}{\pi} \int_{\omega}^{\infty} \left| W(j\omega) \right|^2 \left| T(j\omega; \delta) \right|^2 \Phi_v(\omega) \, d\omega} \]

\( \Phi_v(\omega) \): power spectral density of wind velocity
\( T(j\omega; \delta) \): aircraft transfer function (with uncertain parameters \( \delta \))
\( W(j\omega) \): comfort filter

▷ Current industrial practice: gridding of the parameter space + numerical integration in desired frequency range (*lower bound* to worst-case comfort performance)

▷ Contribution: **robust finite-frequency \( H_2 \) analysis** (*upper bound* to worst-case comfort performance)
Robust $\mathcal{H}_2$ analysis

Vaste literature (Paganini, Feron, Stoorvogel, Iwasaki, Sznaier, ...)

Both time-domain and frequency-domain techniques

Frequency-domain techniques provide an upper bound to the system frequency gains, for all frequencies $\omega$ and admissible uncertainties $\delta$

→ Need tools to compute finite-frequency $\mathcal{H}_2$ norm
Finite-frequency $\mathcal{H}_2$ norm

Given the system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bw(t) \\
z(t) &= Cx(t)
\end{aligned}
$$

with transfer function $G(s)$, the finite-frequency $\mathcal{H}_2$ norm is defined as

$$
\|G(j\omega)\|_{2,\bar{\omega}}^2 = \frac{1}{2\pi} \int_{-\bar{\omega}}^{\bar{\omega}} \text{Tr} \left\{ G(j\nu)^* G(j\nu) \right\} d\nu = \text{Tr} \left\{ B^T W_o(\bar{\omega}) B \right\}
$$

where

$$
W_o(\bar{\omega}) = \frac{1}{2\pi} \int_{-\bar{\omega}}^{\bar{\omega}} (j\nu I - A)^{-*} C^T C (j\nu I - A)^{-1} d\nu.
$$

is the finite-frequency observability Gramian
Finite-frequency observability Gramian

The finite-frequency observability Gramian can be computed as

\[ W_o(\bar{\omega}) = L(\bar{\omega})^* W_o + W_o L(\bar{\omega}), \]

where \( W_o \) is the standard observability Gramian and

\[ L(\bar{\omega}) = \frac{j}{2\pi} \ln \left[ (A + j\bar{\omega}I)(A - j\bar{\omega}I)^{-1} \right]. \]

Main idea: use the above result together with standard robust \( \mathcal{H}_2 \) theory to provide an upper bound to the robust finite frequency \( \mathcal{H}_2 \) norm of a system with parametric uncertainty
Problem formulation

LFR system:

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_q q(t) + B_w w(t) \\
p(t) &= C_p x(t) + D_{pq} q(t) \\
z(t) &= C_z x(t) + D_{zq} q(t) \\
q(t) &= \Delta(\delta)p(t)
\end{aligned}
\]

\(\Delta(\delta) = \text{diag}(\delta_1 I_{s_1}, \ldots, \delta_n I_{s_n}) \in \Delta := \{\Delta(\delta) : -1 \leq \delta_i \leq 1\}\)

\[
M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{bmatrix} A & B_q & B_w \\ C_p & D_{pq} & 0 \\ C_z & D_{zq} & 0 \end{bmatrix}
\]

\(S(M; \Delta) = M_{22} + M_{21} \Delta(I - M_{11} \Delta)^{-1} M_{12}\)

The robust finite-frequency \(\mathcal{H}_2\) norm of the system is defined as

\[
\sup_{\Delta \in \Delta} \|S(M; \Delta)\|_{2, \tilde{\omega}}^2 = \frac{1}{2\pi} \sup_{\Delta \in \Delta} \int_{-\tilde{\omega}}^{\tilde{\omega}} \text{Tr} \left\{S(M; \Delta)^* S(M; \Delta)\right\} d\nu
\]
**Robust $\mathcal{H}_2$ theory**

The system $S(M, \Delta)$ has robust $\mathcal{H}_2$ norm less than $\gamma^2$ if there exist Hermitian matrices $W_o, P_+, P_- > 0, X > 0$ ($X$ commuting with $\Delta$), satisfying

\[
\begin{bmatrix}
    AP_- + P_- A^T + B_q X B_q^T & P_- C^T + B_q X D^T \\
    C P_- + D X B_q^T & D X D^T - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
    AP_+ + P_+ A^T + B_q X B_q^T & P_+ C^T + B_q X D^T \\
    C P_+ + D X B_q^T & D X D^T - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}
\end{bmatrix} < 0 \quad (SDP)
\]

\[
\begin{bmatrix}
    W_o & I \\
    I & P_+ - P_- \end{bmatrix} > 0
\]

\[
\text{Tr} \{B_w^T W_o B_w\} < \gamma^2
\]

[Pagani, ACC’97]
Main result

Let $W_0$, $P_+$, $P_-$, $X$, be the solution of $(SDP)$.

Then, an upper bound to the robust finite-frequency $\mathcal{H}_2$ norm is given by

$$\sup_{\Delta \in \Delta} \| S(M; \Delta) \| _{2,\bar{\omega}}^2 \leq \text{Tr} \left\{ B_w^T (W_0 \mathcal{L}(\bar{\omega}) + \mathcal{L}(\bar{\omega})^* W_0) B_w \right\}$$

where

$$\mathcal{L}(\bar{\omega}) = \frac{j}{2\pi} \ln \left[ (A + j\bar{\omega}I)(A - j\bar{\omega}I)^{-1} \right]$$

with

$$A = A + (P_- C^T + B_q XD^T) R^{-1} C$$

and

$$R = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} - DXD^T$$
Sketch of the proof

- From standard robust $\mathcal{H}_2$, $(SDP)$ guarantees that there exists $Y(j\omega)$ such that

$$S(M, \Delta)(j\omega)^*S(M, \Delta)(j\omega) \leq Y(j\omega), \ \forall \omega, \ \forall \Delta \in \Delta$$

- $Y(j\omega)$ admits a spectral factorization $Y = (N^{-1}M_2)^*(N^{-1}M_2)$, whose state space realization is given by

$$N^{-1}M_2 = \begin{bmatrix}
A + (P - C^T + B_q X D^T)R^{-1}C & B_w \\
R^{-\frac{1}{2}}C & 0
\end{bmatrix}$$

and $W_o$ is its (standard) observability Gramian

- The finite-frequency observability Gramian of the spectral factor $N^{-1}M_2$ is given by $W_o \mathcal{L}(\bar{\omega}) + \mathcal{L}(\bar{\omega})^*W_o$
Extension to dynamic scaling

Choosing constant scaling matrices $X$ in $\text{(SDP)}$ leads in general to conservative upper bounds

Standard robust $\mathcal{H}_2$ theory exploits dynamic scaling matrices of the form

$$X(s) = \begin{bmatrix} C_\psi (sI - A_\psi)^{-1} & I \end{bmatrix} X \begin{bmatrix} [C_\psi (sI - A_\psi)^{-1} & I] \end{bmatrix}^*$$

An upper bound to the robust finite-frequency $\mathcal{H}_2$ norm is obtained by solving a slightly more involved SDP (details in CDC 2010 paper)
**Numerical example**

\[
A = \begin{bmatrix}
-2.5 & 0.5 & 0 & -50 & 0 \\
0 & -1 & 0.5 & 0 & 0 \\
0 & -0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & 100 \\
0 & 0 & 0 & -100 & 0 \\
\end{bmatrix}
\]

\[
B_q = \begin{bmatrix}
0.25 & -0.5 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

\[
B_w = \begin{bmatrix}
0 \\
5 \\
0 \\
0 \\
5 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
C_p \\
C_z
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
D_{pq} \\
D_{zq}
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
\Delta(\delta) = \delta I_2, \quad -1 \leq \delta \leq 1
\]

Exact robust $\mathcal{H}_2$ norm: $\gamma^2 = 1.5311$, attained for $\delta = 0.25$

Exercise: compute the robust finite-frequency $\mathcal{H}_2$ norm for $\overline{\omega} = 50 \text{ rad/s}$
(true value: 0.8919)
Gain plots for different values of $\delta$

Finite-frequency $\mathcal{H}_2$ norm for different values of $\bar{\omega}$ and $\delta$
Robust finite-frequency $\mathcal{H}_2$ norm
Comparison with low-pass filtering

An approximation of the robust finite-frequency $\mathcal{H}_2$ norm can be obtained by cascading the system with a low-pass filter $F(s) = \frac{\tilde{p}^m}{(s+\tilde{p})^m}$ and then computing the standard robust $\mathcal{H}_2$ norm of the resulting system.

<table>
<thead>
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<th>$n_\psi = 0$</th>
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<th>3</th>
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Robust $\mathcal{H}_2$ norm for the system with low-pass filter

<table>
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<td>1.2631</td>
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Robust finite-frequency $\mathcal{H}_2$ norm
**CFCL application**

Model of a civil aircraft including both rigid and flexible body dynamics
\(\delta\): fuel tanks level, normalized in the range \([-1, 1]\)

Resulting uncertain system (including approximated Von Karman filter modeling the wind spectrum and output filters for the specified position):
LFR with 21 states and \(\Delta\) block of size 14

Model derived in the frequency range between 0 and 15 rad/s (no physical meaning outside this range)

*Proposed solution: robust finite-frequency \(H_2\) norm with \(\bar{\omega} = 15\) rad/s, computed in \(N_p\) partitions of the uncertainty interval \([-1, 1]\) (constant scaling)*
Gain plots for different values of $\delta$
Robust finite-frequency $\mathcal{H}_2$ analysis (pos. #12)
Robust finite-frequency $\mathcal{H}_2$ analysis ($N_p = 20$, pos. #4)
Final remarks

Proposed techniques and relaxations managed to solve most COFCLUO clearance problems.

Lyapunov techniques can handle mixed LTI/LTV uncertainties, but are time consuming.

Relaxations allow one to address the key trade off between performance and computational burden.

No a priori hierarchy among the relaxations.

Good matching with worst-case analysis.
Ongoing work

Combine techniques for LTI uncertainties with recent work on memoryless nonlinearities to cope with saturations and rate limiters in actuators (IFAC talk FrB06.1)

Comfort: alternative approach based on $\mu$-analysis and adaptive partitioning of frequency domain (preliminary results in IFAC talk TuA06.2)

Feedback form industry
Those who are interested can find much more in:

Optimization Based Clearance of Flight Control Laws
A. Varga, A. Hansson, G. Puyou (Eds.)

............................................................thanks!