Hybrid Local Search for Constrained Financial Portfolio Selection Problems

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Abstract. Portfolio selection is a relevant problem arising in finance and economics. While its basic formulations can be efficiently solved through linear or quadratic programming, its more practical and realistic variants, which include various kinds of constraints and objectives, have in many cases to be tackled by approximate algorithms. In this work, we present a hybrid technique that combines a local search, as *master* solver, with a quadratic programming procedure, as *slave* solver. Experimental results show that the approach is very promising and achieves results comparable with, or superior to, the state of the art solvers.

1 Introduction

The *portfolio selection* problem consists in selecting a portfolio of *assets* that provides the investor a given expected return and minimises the *risk*. One of the main contributions in this problem is the seminal work by Markowitz [25], who introduced the so-called *mean-variance* model, which takes the variance of the portfolio as the measure of investor's risk. According to Markowitz, the portfolio selection problem can be formulated as an optimisation problem over real-valued variables with a quadratic objective function and linear constraints.

In this paper we consider the basic objective function introduced by Markowitz, and we take into account two additional constraints: the *cardinality* constraint, which limits the number of assets, and the *quantity* constraint, which fixes minimal and maximal shares of each asset included in the portfolio. For an overview of the formulations presented in the literature we forward the interested reader to [7].

We devise a hybrid solution based on a local search metaheuristic (see, e.g., [13]) for selecting the assets to be included in the portfolio, which at each step resorts to a quadratic programing (QP) solver for computing the best allocation for the chosen assets. The QP procedure implements the Goldfarb-Idnani dual algorithm [11] for strictly convex quadratic programs.

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The use of a hybrid solver has been (independently) proposed also by Moral-Escudero et al. [26], who make use of genetic algorithms instead of local search for the determination of the discrete variables.

The paper is organised as follows: In Section 2 we introduce the problem formulation and in the following section (3) we succinctly review the most relevant works that describe metaheuristic techniques applied to formulations closely related to the one discussed in this paper. In Section 4 we present our hybrid solver detailing its components and Section 5 collects the results of the experimental analysis we performed. Finally, in Section 6, we draw some conclusions and point out our plans for further work.

2 Problem Definition

Following Markowitz [25], we are given a set of n *assets*, $A = \{a_1, \ldots, a_n\}$. Each asset a_i has an associated real-valued *expected return* (per period) r_i, and each pair of assets $\langle a_i, a_j \rangle$ has a real-valued *covariance* σ_{ij}. The matrix $\sigma_{n \times n}$ is symmetric and the diagonal elements σ_{ii} represent the *variance* of assets a_i. A positive value R represents the desired expected return. The values r_i and σ_{ij} are usually estimated from past data and are relative [to] one fixed period of time.

A portfolio is a vector of real values $X = \{x_1, \ldots, x_n\}$ such that each x_i represents the fraction invested in the asset a_i. The value $\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$ represents the variance of the portfolio, and is considered as the measure of the risk associated with the portfolio. Whilst the initial formulation by Markowitz [25] was a bi-objective optimisation problem, in many contexts financial operators prefer to tackle a single-objective version, in which the problem is to minimise the overall variance, ensuring the expected return R. The formulation of the basic (unconstrained) problem is thus the following.

$$\min f(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

$$s.t. \quad \sum_{i=1}^{n} r_i x_i \geq R \qquad (1)$$

$$\sum_{i=1}^{n} x_i = 1 \qquad (2)$$

$$0 \leq x_i \leq 1 \quad (i = 1, \ldots, n) \qquad (3)$$

This is a quadratic programming problem, and nowadays it can be solved optimally using available tools despite the NP-completeness of the underlying decision problem [20].

Since R can be considered a parameter of the problem, solvers are usually compared over a set of instances, each with a specific value of minimum required expected return. By solving the problem as a function of R, ranging over a finite

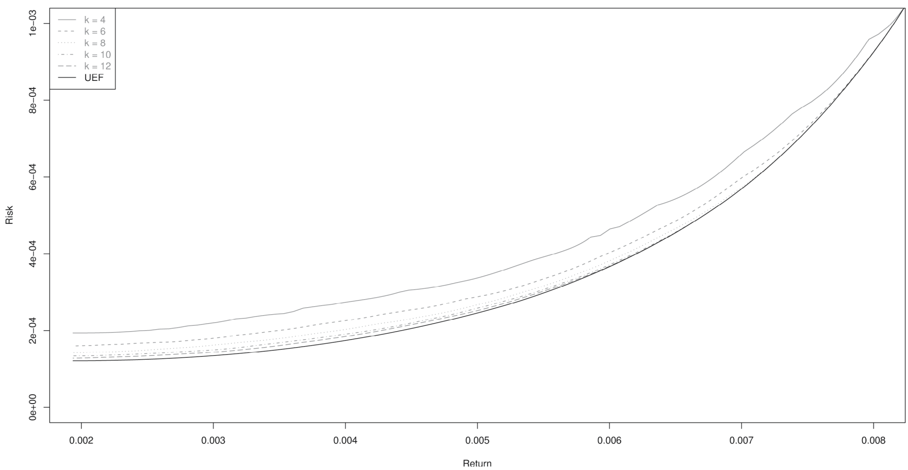


Fig. 1. Unconstrained and constrained efficient frontier

and discrete domain, we obtain the so-called *unconstrained (Pareto) efficient frontier* (UEF), that gives for each expected return the minimum associated risk. The UEF for one of the benchmark instances employed in this study is provided in Figure 1 (the lowest black solid line).

Although the classical Markowitz's model is extremely useful from the theoretical point of view, dealing with real-world financial markets imposes some additional constraints that are going to be considered in this work. In order to model them correctly, we need to add to the formulation a vector of n binary decision variables Z such that $z_i = 1$ if and only if asset i is in the solution (i.e., $x_i > 0$).

Cardinality constraint: The number of assets that compose the portfolio is bounded: we give two values k_{min} and k_{max} (with $1 \leq k_{min} \leq k_{max} \leq n$) such that:

$$k_{min} \leq \sum_{i=1}^{n} z_i \leq k_{max} \tag{4}$$

Quantity constraints: The quantity of each asset i that is included in the portfolio is limited within a given interval: we give a minimum ϵ_i and a maximum δ_i for each asset i, such that:

$$x_i = 0 \quad \vee \quad \epsilon_i \leq x_i \leq \delta_i \qquad (i = 1, \dots, n) \tag{5}$$

Notice that the minimum cardinality constraints are especially meaningful in presence of constraints on the minimum quantity, otherwise they can be satisfied by infinitesimal quantities.

We call CEF the analogous of the UEF for the constrained problem. In Figure 1 we plot the CEF found by our solver for the values $\epsilon_i = 0.01$, $\delta_i = 1$ (for

$i = 1, \ldots, n$), $k_{min} = 1$, and k_{max} varying from 4 to 12. For higher values of k_{max} the cardinality constraint reduces its effect and the curve is almost indistinguishable from the UEF, indeed the distance among the CEF and the UEF[1] becomes smaller than 10^{-3} for the instance at hand.

Constraints 4 and 5 make it intractable to solve real-world instances of the problem with proof of optimality [14]. Therefore, either simplified models are considered, such as formulations with linear objective function [21, 22], or approximate methods are applied.

3 Related Work

Local search approaches have been widely applied to portfolio selection problems under many different formulations. The first work on this subject appearing in the literature is due to Rolland [30], who presents an implementation of Tabu Search to tackle the unconstrained formulation. This formulation is considered also in the implementation of evolutionary techniques in [2, 18, 19]. The use of local search techniques for the constrained portfolio selection problem has been proposed by several authors, including Chang et al. [4], Gilli and Këllezi [9] and Schaerf [31].

The cited works however use local search as a *monolithic* solver, exploring a search space composed of both continuous and discrete variables. Conversely, our hybrid solver focuses on the discrete variables, leaving the determination of the continuous ones to the QP solver. In addition, we consider here a more general problem w.r.t. the cited three papers, including also the possibility to specify a minimum number of assets (and not only the maximum).

Among the population-based methods developed for tackling the constrained formulation, we mention Streichert et al. [33], in which the cardinality constrained variant is considered, and memetic algorithm approaches introduced in [12, 15, 24]. These strategies, by being inherently effective in diversifying the search, exhibit good performance especially in multi-objective formulations, as shown by the family of Multi-Objective Evolutionary Algorithms [17, 8, 27, 33]. Finally, Ant Colony Optimisation has also been successfully applied to portfolio problems modelled with the cardinality constraint in [1, 23].

For the sake of completeness, we also mention interesting hybrid heuristic techniques based on linear programming that have been introduced in [32] and deal with a linear objective function formulation with integer variable domains. In this case, the value assigned to a variable represents the actual amount invested in the asset. The basic idea behind these approaches is to relax the discrete constraint on quantities, transforming the problem into a linear programming problem and find a solution to it. Fractional asset weights are then rounded to the closest admissible discrete quantity and a possible infeasible solution is repaired heuristically. More robust strategies use the solution to the continuous relaxation to feed a mixed integer-linear programming solver [16, 20].

[1] Measured by what we call average percentage loss, introduced in Section 5.1.

4 A Hybrid Local Search Solver for Portfolio Selection

Our master solver is based on local search, which works on the space induced by the vector Z only. For computing the actual quantities X, it invokes the QP (slave) solver, using as the input assets only those such that $z_i = 1$ in the current state.

In order to apply local search techniques we need to define the search space, the cost function, the neighbourhood structures, and the selection rule for the initial solution.

4.1 Search Space and Cost Function

The search space is composed of the all 2^n possible configurations of Z, with the exception of assignments that do not satisfy Constraints (4). These constraints are therefore implicitly enforced by the local search solver by excluding them from the search space. On the contrary, states that violate Constraints (1), (2), (3), or (5) are included, and these constraints are passed to the QP solver that handles them explicitly.

The QP solver receives as input only those assets included in the state under consideration, and it produces the assignment of values to the corresponding x_i variables. For all assets a_i that are not included in the state we obviously set $x_i = 0$. In addition, the QP solver also returns the computed risk f for the solution produced, which represents the cost of the state.

If the QP solver is unable to produce a feasible solution it returns the special value $f = +\infty$ (and the values x_i returned are not meaningful). In this case, we relax Constraint (1) and we build the configuration, using only the assets included that gives the highest return without violating the other constraints. This construction is done by a greedy algorithm that sorts the assets by the expected return and assigns the maximum quantity to each asset in turn, as long as the sum is smaller than 1.

In the latter case the cost is the degree of violation of Constraint (1) multiplied by a suitably large constant (that ensures that return related costs are always bigger than risk related ones).

4.2 Neighbourhood Structure

The neighbourhood relation we propose is based on addition, deletion and replacement of an asset. A move m is identified by a pair $\langle i, j \rangle$, where a_i is the asset to be added and a_j is the asset to be deleted ($i, j \in \{1, \ldots, n\}$). The value of i can also be 0, meaning that no asset is added. Analogously for j, if $j = 0$ it means that no asset is deleted.

Notice that not all pairs $m = \langle i, j \rangle$, with $i, j \in \{0, 1, \ldots, n\}$, correspond to a feasible move, since some values are meaningless, e.g. inserting an asset already present or setting both i and j to zero (*null move*). Moreover, moves that violates Constraints (4) are also considered infeasible, e.g. a delete move when the number of assets is equal to the minimum.

4.3 Initial Solution Construction

For the initial solution, we use three different strategies, that are employed at different stages of the search (as explained in Section 4.4). For all three, we ensure that Constraints (4) are always satisfied.

RandomCard: We draw at random a number k (between k_{min} and k_{max}), and we insert k randomly selected assets.

MaxReturn: We build the portfolio that produces the maximum possible return (independently of the risk)

PreviousPoint: We use the final solution of the previously computed point of the frontier

4.4 Local Search Techniques

We implemented three local search techniques, namely *Steepest Descent* (SD), *First Descent* (FD), and *Tabu Search* (TS).

The SD strategy relies on the exhaustive exploration of the neighbourhood and the selection of the neighbour that has the minimal value of f (breaking ties at random). The SD strategy stops as soon as no improving move is available, i.e., when a local minimum has been reached. FD behaves as SD with the difference that, as soon as an improving move is found, it is selected and the exploration of the current neighbourhood is interrupted.

For TS we use a dynamic-size tabu list to implement a short term prohibition mechanism and the standard aspiration criterion [10]. Like for SD, we search for the next state by exploring the full neighbourhood (excluding infeasible moves) at each iteration.

In order to make the solvers more robust, for all techniques, we make two runs for each value of R: one using the RandomCard initial solution construction, and the other one starting from the best of the previous point (PreviousPoint initial solution). For the very first point of the frontier (highest requested return and no previous point available) we use instead the MaxReturn construction.

5 Experimental Analysis

In this section, we first present the benchmark instances and the settings of our solver. In the following subsections, we show the comparison with all the previous works that use the same formulation. We conclude showing a search space analysis that tries to explain the behaviour of our solvers on the proposed instances.

5.1 Benchmark Instances

We experimented our techniques on two groups of instances obtained from real stock markets and used in previous works. The first is a group of five instances taken from the repository `ORlib` available at the URL `http://mscmga.ms.ai.`

Table 1. The benchmark instances

Inst.	Origin	assets	UEF
Group 1			
1	Hong Kong	31	$1.55936 \cdot 10^{-3}$
2	Germany	85	$0.412213 \cdot 10^{-3}$
3	UK	89	$0.454259 \cdot 10^{-3}$
4	USA	98	$0.502038 \cdot 10^{-3}$
5	Japan	225	$0.458285 \cdot 10^{-3}$
Group 2			
S1	USA (DataStream)	20	4.812528
S2	USA (DataStream)	30	8.892189
S3	USA (DataStream)	151	8.64933

`ac.uk/ jeb/orlib/portfolio.html` . These instances have been proposed by Chang et al. [4] and have been studied also in [1, 26, 31]. The second group of three instances have been provided to us by M. Schyns and are used in [5].

For the first group, a discretised UEF composed of 100 equally distributed values for the expected return R is provided along with the data. For the second group, we computed the discretised UEF ourselves using the QP solver with all assets available and no additional constraints.

As in previous works, we evaluate the quality of our solutions employing an aggregate indicator that measures the deviation of the CEF found by the algorithms w.r.t. the UEF on the whole set of frontier points. We call this measure *average percentage loss* (*apl*) and we define it as follows: let R_l be the expected return, $V(R_l)$ and $V_U(R_l)$ the values of the function f returned by the solver and the risk on the UEF, respectively, and $l = 1, \ldots, p$ where p is the number of points of the frontier; the average percentage loss is equal to $\frac{100}{p} \sum_{l=1}^{p} (V(R_l) - V_U(R_l))/V_U(R_l)$. Table 1 illustrates for all instances the original market, and the average variance of the UEF.

5.2 Experimental Setting of the Solvers

Experiments were performed on an Apple iMac computer equipped with an Intel Core 2 Duo (2.16 GHz) processor and running Mac OS X 10.4; the SD, FD and TS metaheuristics have been coded in C++ exploiting the framework EasyLocal++ [6], the QP solver has also been coded in C++ and is made publicly available from one of the authors' website[2]. The executables were obtained using the GNU C/C++ compiler (v. 4.0.1).

Concerning the algorithms setting, SD and FD have no parameter to be set; for TS we tuned its parameters by means of a statistical technique called F-race [3] and found that the algorithm is very robust with respect to parameter setting. We set the tabu list size in the range $[3 \ldots 10]$ and we stop the execution of TS when a maximum of 100 iterations without improvement was reached.

[2] `http://www.diegm.uniud.it/digaspero/`

5.3 Comparison with Previous Results

Due to the different formulations employed by the authors, the only papers we can compare with are those of Schaerf [31] and Moral-Escudero et al. [26], who employ the same set of constraints on the `ORlib` instances, and with Crama and Schyns [5] who deal with a slightly different setting and with a novel set of instances. Concerning Chang et al. [4], as already pointed out in [31], even though they work on the `ORlib` instances (and with the same constraints), a fair comparison with their solutions is not possible because the problem is solved by taking points along the frontier that are not homogeneously distributed.

Armañanzas and Lozano [1] work on a variant of the problem for which the values k_{min} and k_{max} coincide (i.e., $k_{min} = k_{max} = K$) on the `ORlib` instances. However, due to what we believe is an error in the implementation of their solution methods [3] they obtain a set of points that are infeasible w.r.t. Constraint (2). In details, they assign to the assets i for which $z_i = 1$ chosen by their ACO algorithm the quantity $x_i = (\delta_i - \epsilon_i)/K$, therefore since they set $\epsilon_i = 0.001, \delta_i = 1$ for all $i = 1, \ldots, n$, they obtain $\sum_{i=1}^{n} x_i = 0.999$ instead of 1. For this reason we could not compare our solvers with [1], nevertheless we are going to present some results on the behaviour of one of our solvers on the formulation proposed in that paper.

Comparison with Schaerf [31] and Moral-Escudero et al. [26]. For this comparison, we set the constraint values exactly as in [26, 31]: $\epsilon_i = 0.01$ and $\delta_i = 1$ for $i = 1, \ldots, n$, and $k_{max} = 10$ for all instances. The minimum cardinality is not considered in the cited work, and therefore we set it to $k_{min} = 1$ (i.e., no limitation).

Table 2 shows best results and running times obtained by our three solvers in comparison with previous work. Since Moral-Escudero et al. [26] report only the best outcomes of their solvers, in order to fairly compare with them we have to present the results as the *minimum* average percentage loss w.r.t. the UEF found by the algorithm.

The results of our solvers are the best CEFs found in 30 trials of the algorithm on each instance and the running times reported are those of the best trial (exactly as in [26]). Running times of [31] are obtained re-running Schaerf's software on our machine, those of Moral-Escudero et al. are taken from their paper, and are obtained using a PC having about the same performances.

Table 2 shows that we obtain results superior to [31] both in terms of risk and running times. This suggests that the hybrid solver outperforms monolithic local search ones. Regarding [26], we obtain with SD exactly the same results of their best solver, but in a much shorter time (on a comparable machine).

As already pointed out in [31], even though Chang et al. [4] solve the same instances (and with the same constraints), a fair comparison with their solutions is not possible. This is because they consider the CEF differently. Specifically, they do not solve a different instance for each value of R, but (following Perold [28]), they reformulate the problem without Constraint (1) and with the following

[3] We found the error in our analysis of the data provided to us by J. Lozano.

Table 2. Comparison of results with Schaerf [31] and Moral-Escudero et al. [26]

	FD + QP		SD + QP		TS + QP		GA + QP [26]		TS [31]	
Inst.	min apl	time	min apl	time	min apl	time	min apl	time	min apl	time
1	0.00366	1.3s	0.00321	4.3s	0.00321	17.2s	0.00321	415.1s	0.00409	251s
2	2.66104	5.3s	2.53139	20.3s	2.53139	61.3s	2.53180	552.7s	2.53617	531s
3	2.00146	5.4s	1.92146	23.6s	1.92133	69.5s	1.92150	886.3s	1.92597	583s
4	4.77157	7.6s	4.69371	27.6s	4.69371	80.0s	4.69507	1163.7s	4.69816	713s
5	0.24176	15.7s	0.20219	69.5s	0.20210	210.7s	0.20198	1465.8s	0.20258	1603s

objective function: $f(X) = \lambda f_1(X) + (1 - \lambda) f_2(X)$. The problem is then solved for different values of λ, and what they obtain is the solution for a set of values for R which are not homogeneously distributed.

Comparison with Crama and Schyns [5]. Since the results of Crama and Schyns [5] are presented in graphical form and make use of a slightly different cost function (i.e., they consider the standard deviation instead of the variance as the risk measure) we re-run their solver[4] on the three instances employed in their experimentation employing the same parameter setting reported in their paper. The constraints set in this experiment are as follows: $k_{min} = 1$, $k_{max} = 10$, $\epsilon_i = 0$, and $\delta_i = 0.25$.

In Table 3 we present the outcome of this comparison. For each algorithm we report in three columns the average and the standard deviation (in parentheses) of the average percentage loss w.r.t. the UEF, and the average time spent by the algorithm. The data was collected by running 30 times each algorithm on each instance and computing the whole CEF.

From the table it is clear that, in terms of solution quality, the family of our solvers outperforms the SA approach of Crama and Schyns. Looking at the times, we can see that SA, in general, exhibits shorter running times than our hybrid SD and TS approaches. This can be explained by the strategy employed by both our algorithms that thoroughly explore the full neighbourhood of each solution whereas the SA randomly picks out only some neighbours thus saving time in the evaluation of the cost function. Moreover, the slave QP procedure is more time-consuming than the solution evaluation carried out by Crama and Schyns, however it allows us a higher accuracy on the assignment of the assets.

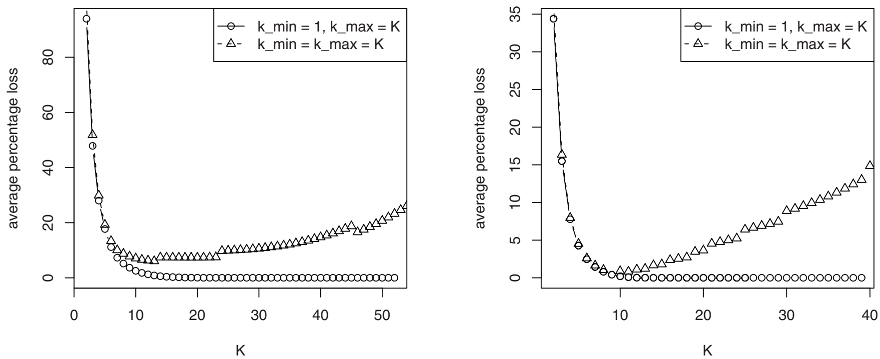
Results for fixed cardinality portfolios. As mentioned previously, even though we cannot compare our results with the work of Armañanzas and Lozano [1], we decided to show some results of the SD solver on the `ORlib` instances by setting the cardinality constraints so that they force the constructed portfolio to have exactly $k_{min} = k_{max} = K$ assets as in [1]. The quantity constraints are set as in the first set of experiments, i.e. $\epsilon_i = 0.01$, and $\delta_i = 1$.

In Figure 2 we plot the behaviour of the average percentage loss found by our SD + QP solver at different values of K on a selected pair of instances. The

[4] The executable was kindly provided to us by M. Schyns.

Table 3. Comparison of results with Crama and Schyns [5]

Inst.	FD + QP apl	FD + QP time	SD + QP apl	SD + QP time	TS + QP apl	TS + QP time	SA [5] apl	SA [5] time
S1	0.72 (0.094)	0.3s	0.35 (0.0)	1.4s	0.35 (0.0)	4.6s	1.13 (0.13)	3.2s
S2	1.79 (0.22)	0.5s	1.48 (0.0)	3.1s	1.48 (0.0)	8.5s	3.46 (0.17)	5.4s
S3	10.50 (0.51)	10.2s	8.87 (0.003)	53.3s	8.87 (0.0003)	124.3s	16.12 (0.43)	30.1s



(a) Results on instance 2. **(b)** Results on instance 5.

Fig. 2. Average percentage loss found by our SD + QP solver varying K

curve is compared with the average percentage loss computed by the same solver but relaxing the minimum cardinality constraint to $k_{min} = 1$ (i.e., just allowing to include an increasing number of assets in the portfolios, but not obliging the solver to compel to a fixed cardinality).

From the pictures we can notice an interesting phenomenon: the two curves are almost indistinguishable up to a value of K for which the fixed cardinality solutions tend to have an higher average percentage loss. In a sense, this sort of *minimum* represent the best compromise in the cardinality, i.e., the optimal fixed number of assets K that minimises the deviation from the best achievable returns (i.e, the UEF values).

5.4 Search Space Analysis

We study the search space main characteristics of the instances composing the benchmarks with the aim of providing an explanation for the observed algorithm behaviour and elaborating some guidelines for understanding the hardness of an instance when tackled with our hybrid local search. Once cardinality constraints are set, in general we are interested in studying the characteristics of the search space of the single instances of the problem along the frontier, i.e., at fixed values of return R. Among the 100 points composing the frontier, we took five samples homogeneously distributed along the frontier, in order to estimate the characteristics of the search spaces encountered by our solver along the whole frontier. Moreover, constrained instances with different values of k_{min} and k_{max} have been considered. In our experiments, we chose $(k_{min}, k_{max}) \in \{(3,3),(6,6),(10,10),(1,3),(1,6),(1,10)\}$.

One of the most relevant search space characteristics is the number of global and local minima. The number of local minima is usually taken as an estimation of the ruggedness of the search space, that, in turn, is roughly negatively correlated with local search performance [13]. In order to estimate the number of minima in an instance, we run a deterministic version of SD (called SD_{det})[5] starting from initial states either produced by complete enumeration (for very small size instances) or by uniformly sampling the search space.

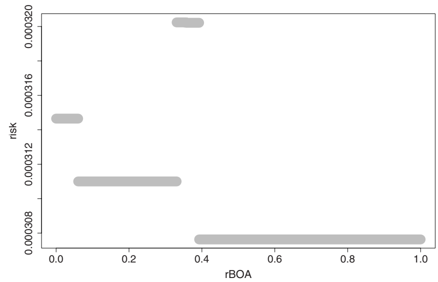
Our analysis shows that the instances of the benchmarks have a very small number of local minima, and only one global minimum (i.e., either a certified global minimum, when exhaustive enumeration is performed, or the best known solution, otherwise). Most of the analysed instances have only one minimum and the other instances have not more than six minima. We observed that the latter cases occur usually at low values of return R. Instance 4 is the one with the greatest number of local minima, while the remaining instances have very few cases with local minima.

This analysis may provide an explanation for the very similar performance exhibited by SD and TS in terms of solution quality. To strengthen this argument, we also studied global and local minima basins of attraction, in order to estimate the probability of reaching a global minimum [29]. Given a deterministic algorithm such as SD_{det}, the basin of attraction $\mathcal{B}(\overline{s})$ of a minimum $\overline{s}$, is defined as the set of states that, taken as initial states, give origin to trajectories that ends at point $\overline{s}$. The cardinality of $\mathcal{B}(\overline{s})$ represents its size (in this context, we always deal with finite spaces). The quantity $rBOA(\overline{s})$, defined as the ratio between the size of $\mathcal{B}(\overline{s})$ and the search space size, is an estimation of the reachability of state $\overline{s}$. If the initial solution is chosen at random, the probability of finding a global optimum s^* is exactly equal to $rBOA(s^*)$. Therefore, the higher is this ratio, the higher is the probability of success of the algorithm. Both SD and TS incorporate stochastic decision mechanisms and TS is also able to escape from local minima, therefore the estimation of basins of attraction size related to SD_{det} provides a lower bound on the probability of reaching the global optimum when using SD and TS.

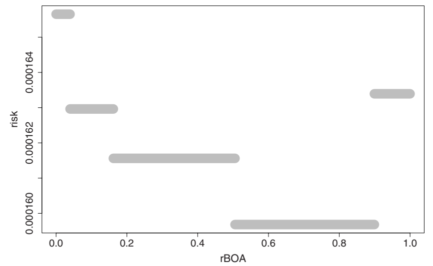
The outcome of our analysis is that global minima have usually a quite large basin of attraction. Representative examples of these results are depicted in Figures 3a, 3b, 3c and 3d; segments represent the basins of attraction: their length corresponds to $rBOA$ and their y-value is the objective value of the corresponding minimum. We can note that global minima have a quite large basin of attraction whose $rBOA$ ranges from 30% (in Figure 3c) to 60% (in Figure 3a).

It is worth remarking that these large basins are specific for our hybrid solver, and this is not the case for monolithic local search ones. The presence of large basins of attraction for the global optimum suggests that the best strategy for

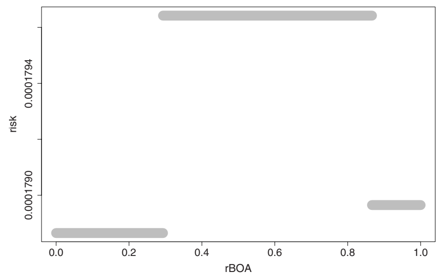
[5] Ties are broken by enforcing a lexicographic order of states.



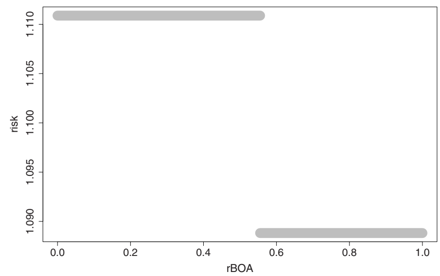
(a) Instance 4: $k_{min} = k_{max} = 3$, $R = 0.0037524115$.



(b) Instance 4: $k_{min} = 1$, $k_{max} = 6$, $R = 0.0019368822$.



(c) Instance 4: $k_{min} = 1$, $k_{max} = 10$, $R = 0.0037524115$.



(d) Instance S3: $k_{min} = 1$, $k_{max} = 6$, $R = 0.260588$.

Fig. 3. Basins of attraction of minima on two benchmark instances with different cardinality and return constraints

tackling these instances is simply to run SD with random restarts, and that there is no need for a more sophisticated solver such as TS.

However, since TS has better exploration capabilities than SD, it could still show superior performances on other, possibly more constrained, instances. Indeed, it is possible construct artificial instances with a large number of local minima and a small basin for the global one; it is straightforward to show that for such instances TS performs much better than SD for all values of R.

6 Conclusions and Future Work

Experiments show that our solver is comparable with (or superior to) the state of the art for the less constrained problem formulation (no minimum). Comparison for the general problem are subject of ongoing work. In the future, we plan to adapt this approach to tackle other formulations, such as the discrete formulation that is particularly interesting for some investors. This formulation enables us to take into account aspects of real-world finance, such as transaction costs. To this extent, instances including minimum lots will be investigated, since assets generally cannot be purchased in any quantity and the amount of money to be invested in a single asset must be a multiple of a given minimum lot [20]. Moreover, we are going to include also asset preassignments, that will be useful for representing investor's subjective preferences.

We also aim at identifying difficult instances and verify whether more sophisticated local search metaheuristics, such as TS, could improve on the results of the simple SD strategy.

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