Plane Magnetic Field Analysis with the Finite Formulation

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Abstract: The paper describes the solution of the static and quasi-static bi-dimensional magnetic field problems in the framework of the Finite Formulation. The Finite Formulation is a numerical approach, consisting of an algebraic formulation of the field laws. It adopts the global variables together with the primal-dual tessellation of the space and time domains. The critical issue of the derivation of the constitutive equations is also analyzed showing two possible approaches. Some 2D test problems are discussed in the case of non homogeneous and non linear media. An axisymmetric eddy current problem has been analyzed in addition and the obtained results have been compared with those from the well established Finite Elements method.

Index Terms: magnetostatic, eddy-currents, Finite Formulation, dual meshes.

1 Introduction

The numerical methods in computational electromagnetism start from a differential approach based on the Maxwell equations formulated, both in the differential form or in the integral form, in terms of the field quantities \( \rho, J, E, H, B, D \). Using a specific discretization method, like the Finite Differences, the Finite Differences in Time Domain [19], the Finite Elements [1], [3], the Edge Elements [4], or the Finite Integration Techniques [18],[9], a set of algebraic equations can be derived to solve numerically the field equations.

Recently, a relevant theoretical effort has been performed, in order to give a rationale in the numerical treatment of Maxwell’s equations; the work of Bossavit [6], [5] and the Finite Formulation of Tonti [13] belong to this framework. At the base of both the works, even if not explicitely declared in the Tonti’s Finite Formulation, are the mathematical tools of the differential forms [7], [21, pp. 257-293] and of the algebraic topology [20], [17], used to represent the physical entities of the fields. According to these works the laws of electromagnetism can be restated directly as algebraic equations in a unique way in terms of variables like currents \( I \), voltages \( U \) or magnetic fluxes \( \Phi \); according to algebraic topology, these variables are cochains which yield real numbers when they act on chains, and indeed, they can be considered the mathematical abstractions with the closest match to observations we make of the electromagnetic phenomena. To solve the field problems, the constitutive equations must be also considered, which are a mapping between pointwise objects, the fields or the differential forms used to describe them. This mapping is the Hodge Operator and a crucial issue in computational electromagnetism is to approximate its discrete version. [8]. In Finite Formulation these mathematical aspects are masked under the hypothesis of the local uniformity of the fields.
Aim of this paper is to apply and test the Finite Formulation approach tailoring it to the case of static and quasi-static plane magnetic field analyses in the presence of non homogeneous or non linear media. Moreover the crucial point of writing constitutive equations under the hypothesis of local uniformity of the fields is analysed in detail showing two possible approaches.

2 Classification of variables

An important issue of the Finite Formulation is a qualitative analysis of the physical variables which can be grouped in three main classes: the configuration, the source and the energy variables; this classification has been initially introduced in [10], [11] and then revised and improved in [13]. This classification, introduced here for electromagnetic variables, is general and has been applied by Tonti to different fields of physics. The configuration variables describe the configuration of the field or of the system; examples of the configuration variables are: the electric potential \( V \), the electric voltage \( U \), the electric field vector \( \mathbf{E} \), the magnetic flux \( \Phi \). The source variables describe the sources of the field without involving the material parameters; examples of the source variables are: the electric charge flow \( Q_f \), the electric current \( I \), the magnetic voltage \( F \), the magnetic field \( \mathbf{H} \). The energy variables are obtained as the product of a configuration variable by a source variable; examples of the energy variables are: the work, the magnetic energy density, the Poynting vector.

The link between fields (pointwise objects) belonging to the class of configuration or source variables respectively, are the constitutive equations that contain the material properties and the metric notions such as lengths, areas and volumes; in this paper, the magnetic constitutive equation and the Ohm’s law will be examined in detail.

The Finite Formulation makes also use of the so called global variables that formally are \( p \)-cochains, [17]. The global variables are associated with oriented space and time elements like points \( P \), lines \( L \), surfaces \( S \), volumes \( V \), time instants \( I \) and time intervals \( T \) (the bold face evidence that the elements are oriented). The global variables, relevant to our static or quasi-static magnetic field analysis are: the electric potential impulse \( \mathcal{V} \), the electric voltage impulse \( \mathcal{U} \), the magnetic flux \( \Phi \), the circulation \( p \) of the magnetic vector potential, the electric charge flow \( Q_f \), the magnetic voltage impulse \( \mathcal{F} \). These global variables are reported in Table 1 divided into configuration and source variables.

<table>
<thead>
<tr>
<th>Global Variables</th>
<th>Configuration Variables [Wb]</th>
<th>Source Variables [C]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electric Potential Impulse</td>
<td>( \mathcal{V} = \int_T \mathcal{V} dt )</td>
<td>( Q_f = \int_{\tilde{T}\tilde{S}} \mathcal{J} \cdot dS dt )</td>
</tr>
<tr>
<td>Electric Voltage Impulse</td>
<td>( \mathcal{U} = \int_{\tilde{L}L} \mathbf{E} \cdot dL dt )</td>
<td>( \mathcal{F} = \int_{\tilde{T}L} \mathbf{H} \cdot dL dt )</td>
</tr>
<tr>
<td>Magnetic Flux</td>
<td>( \Phi = \int_S \mathbf{B} \cdot dS )</td>
<td></td>
</tr>
<tr>
<td>Circulation</td>
<td>( p = \int_L \mathbf{A} \cdot dL )</td>
<td></td>
</tr>
</tbody>
</table>

Global variables are related to the fields (electric potential \( V \), electric field \( E \), magnetic induction \( B \), magnetic vector potential \( A \), current density \( J \), magnetic field \( H \) respectively) by means of an integration performed on lines \( L \), surfaces \( S \), and time intervals \( T \); a tilde on the integration domains \( \tilde{T}, \tilde{S} \) is used to distinguish one of the two possible orientations of the geometric elements, as will be explained later.

2.1 Orientation of the geometrical elements

Space \( P, L, S, V \) and time elements \( I, T \) need to be oriented in order to define the sign of the associated global variables. There are two kinds of orientations: inner and outer. Moreover the
notion of outer orientation depends on the dimension of the embedding space.

If we consider the 3D space, the inner orientation can be defined as follows: the points \( P \) can be oriented as sinks, the lines \( L \) are oriented by a direction chosen on them, the surfaces \( S \) are endowed with inner orientation when their boundary line (\( \partial S \)) has an inner orientation and the volumes \( V \) are endowed with inner orientation when their boundary surface (\( \partial V \)) has inner orientation.

On the other hand the outer orientation of a volume \( V \) is based on the choice of outward (or inward) normals on its boundary. For a surface \( S \) the outer orientation is defined when we fix an arrow crossing the surface from the negative to the positive face: it is equivalent to the inner orientation of the line \( L \) crossing the surface. A line \( L \) is endowed with outer orientation when a direction of rotation around the line has been defined: it is equivalent to the inner orientation of a surface \( S \) crossing the line. The outer orientation of a point \( P \) can be defined as the inner orientation of the volume \( V \) enclosing the point.

In the case of the time axis, the embedding space is 1D; therefore a primal instant \( I \) is endowed with inner orientation when its point on the time axis is oriented as a sink; a primal interval \( T \) has the inner orientation when it is oriented toward increasing time. The outer orientation of a dual instant \( \tilde{I} \) is the inner orientation of the primal interval to which it belongs. The outer orientation of a dual interval \( \tilde{T} \) is the inner orientation of the primal instant internal to it.

### 2.2 Discretisation of space and time in cell complexes

Instead of considering all the possible oriented geometric elements, a discretisation of space and time is now introduced.

Nodes \( p_k \), edges \( l_j \), faces \( s_i \), and cells \( v_h \) form a cell complex \( G \) and are representative of points \( P \), lines \( L \), surfaces \( S \) and volumes \( V \) respectively; the subscripts are required to number the geometrical elements. Once we have defined a cell complex, called primal complex, we can introduce another cell complex called dual complex \( \tilde{G} \) made of geometrical elements denoted by \( \tilde{p}_h \), \( \tilde{l}_j \), \( \tilde{s}_i \), \( \tilde{v}_k \) with a tilde to distinguish them from the corresponding geometrical elements of the primal cell complex \( G \). The duality between the two complexes \( G \) and \( \tilde{G} \) assures that the geometrical elements correspond as follows: \( p_k \leftrightarrow \tilde{v}_k \), \( l_j \leftrightarrow \tilde{s}_i \), \( s_i \leftrightarrow \tilde{l}_j \), \( v_h \leftrightarrow \tilde{p}_h \).

The primal cell complex considered here is based on Delaunay triangles. Different choices of the point, inside each primal cell \( v_h \), to be chosen as dual point \( \tilde{p}_h \) of the dual cell complex, are possible. In this paper the barycenters of the primal cells are selected as \( \tilde{p}_h \) and therefore a dual edge \( \tilde{l}_j \) connects the barycenters of two adjacent primal cells via the barycenter of the common face \( s_i \), Fig. 1; this is the barycentric subdivision. The \( G \)-\( \tilde{G} \) barycentric cell complexes are easy to be generated and can be straightforwardly extended to the 3D geometries.

In the plane field problems, primal cells are prisms with unit thickness (or unit angular width in the axisymmetric fields) and triangular base; the dual edges are broken lines crossing the primal faces at their barycenter. Dual cells are prisms with unit thickness (or unit angular width in axisymmetric fields) and are staggered respect to the primal cells.

We assume to assign to all the elements of a primal complex \( G \) the inner orientation. If a cell complex \( G \) has been endowed with inner orientation, the outer orientation is induced on the space elements of its dual complex \( \tilde{G} \) and therefore all the dual elements are automatically endowed with outer orientation.

We indicate with \( N_v \) the number of primal cells \( v_h \), with \( N_s \) the number of primal faces \( s_i \) (due to the plane symmetry the top-bottom faces of the prisms are omitted in the count), with \( N_l \) the number of primal edges \( l_j \) (the edges normal to the plane of symmetry are neglected in the count), and with \( N_p \) the number of primal nodes \( p_k \) (only the three vertices of the top triangle of each primal cell are accounted for).

In the case of the dual cell complex \( \tilde{G} \) the duality assures that the following equalities hold: \( N_{\tilde{v}} = N_p \), \( N_{\tilde{s}} = N_l \), \( N_{\tilde{l}} = N_s \) and \( N_{\tilde{p}} = N_{v} \).
Figure 1: Detail of a simplicial primal-dual barycentric cell complex for plane or axisymmetric field problems. Inner and outer orientations of a primal and dual line and of a primal and dual face are shown.

Figure 2: primal - dual discretisation of the time axis.
A primal cell complex is introduced on the time axis also. Its elements are the primal instants \( t_1, t_2, \ldots, t_n \) and the primal intervals \( T_1, T_2, \ldots, T_n \) with inner orientation. The dual complex has dual instants \( \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n \) and dual intervals \( \tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n \), Fig. 2, endowed with outer orientation that is the inner orientation of the primal complex.

### 2.3 Global variables and cell complexes

As a result of Tonti’s Finite Formulation [15], [16], there is a coupling between global variables and oriented space and time geometrical elements of a cell complex, such that: the configuration variables can be associated with space and time elements endowed with inner orientation and so to the primal cell complexes, while the source variables can be associated with space and time elements endowed with outer orientation and so to dual cell complexes. This association of global variables to oriented space and time geometrical elements, plays a key role to provide a discrete formulation of laws in many theories of physics and it is useful in computational electromagnetism; in the case of the magnetostatic and magneto quasi-static field, the association above can be summarised as follows:

- the electric potential impulse \( V[T_n, p_k] \) is relative to primal intervals \( T_n \) and to primal nodes \( p_k \);  
- the electric voltage impulse \( U[T_n, l_j] \) is relative to the primal intervals \( T_n \) and to the primal edges \( l_j \);  
- the magnetic flux \( \Phi[t_n, s_i] \) is relative to the primal instants \( t_n \) and primal faces \( s_i \);  
- the circulation \( p[t_n, l_j] \) of the magnetic vector potential is relative to the primal instants \( t_n \) and to the primal edges \( l_j \);  
- the electric charge flow \( Qf[\tilde{T}_n, \tilde{s}_j] \) is relative to dual intervals \( \tilde{T}_n \) and to dual faces \( \tilde{s}_j \);  
- the magnetic voltage impulse \( F[\tilde{T}_n, \tilde{l}_i] \) is relative to dual intervals \( \tilde{T}_n \) and to dual edges \( \tilde{l}_i \).

### 3 Magnetostatic and magneto quasi-static laws in finite form

In the following, the laws relevant for magnetostatic and magneto quasi-static field analysis will be applied to the corresponding geometrical elements of the above defined cell complexes in space and time; so doing, the corresponding algebraic equations will be derived. These algebraic equations link configuration variables with configuration variables and source variables with source variables and are valid in whatever medium. They do not involve metric notions, i.e. lengths, areas, measures of volumes and durations and they are valid both in the large and in the small, [7], [13].

#### 3.1 Gauss magnetic law

Considering the primal cells \( v_h \) and the primal faces \( s_i \), Fig. 3, the Gauss magnetic law, with matrix notation, becomes:

\[
\mathbf{D} \Phi = 0 \quad (1)
\]

where \( \mathbf{D} \) is the incidence matrix \( \mathbf{D} = ||d_{hi}|| \) of dimension \( N_v \times N_s \) between the inner orientations of \( v_h \) and \( s_i \); \( \Phi = [\Phi_1, \Phi_2, \ldots, \Phi_{N_s}]^T \) is the vector of fluxes of dimension \( N_s \). Due to the plane symmetry, only the lateral faces of the volume \( v_h \) have to be considered in (1). The flux \( \Phi_i \) relative to a primal
Figure 3: A primal volume $v_h$ and the corresponding primal faces exploded with the inner orientation for the case of a plane field; with these orientations $\Phi_1 + \Phi_2 - \Phi_3 = 0$. For a lateral primal face $s_i$ with primal edges $l_a, l_b$ normal to the plane of symmetry $\Phi_i = p_b - p_a$, on the right a dual face $\tilde{s}_j$ (top or bottom of $\tilde{v}_k$) and its boundary of dual edges $l_i$ with outer orientation; with these orientations $\tilde{F}_1 - \tilde{F}_2 - \tilde{F}_3 - \tilde{F}_4 + \tilde{F}_5 + \tilde{F}_6 - \tilde{F}_7 = Q^t$.

face $s_i$, can be expressed as the circulation $p$ of the magnetic vector potential along the boundary edges of the face $s_i$:  

$$\Phi_i = \sum_{j=1}^{4} c_{ij} p[t_n, l_j]$$  

(2)

c_{ij} are the incidence numbers between the inner orientation of the primal edge $l_j$ and the inner orientation of the corresponding primal face $s_i$. Due to the plane symmetry, only the boundary primal edges of $s_i$ normal to the plane of symmetry contribute in (2). Introducing the vector of the circulations $p = [p_1, p_2, ..., p_{N_p}]^T$ along the primal edges $l_j$ normal to the plane of symmetry, of dimension $N_p$, (2) becomes:  

$$\Phi = Cp$$  

(3)

being $C = ||c_{ij}||$ the incidence matrix, of dimension $N_s \times N_p$, (3) identically satisfies (1) being $D \equiv 0$.

### 3.2 Ampère’s law

Assuming here a charge flow normal to the plane of symmetry, the $Q^f$ crossing the lateral dual faces of $\tilde{v}_k$ is null. Therefore, considering the dual faces $\tilde{s}_j$ laying on the symmetry plane, Fig. 3 right, the Ampère law can be written as:  

$$\tilde{C} \mathbf{F} = Q^t$$  

(4)

being $\tilde{C} = ||\tilde{c}_{ji}||$ the incidence matrix, of dimension $N_{\tilde{F}} \times N_{\tilde{l}}$, between the outer orientations of $\tilde{v}_k$ and $l_i$; $\mathbf{F} = [\mathbf{F}_1, \mathbf{F}_2, ..., \mathbf{F}_{N_{\tilde{l}}}]^T$ is the vector of the magnetic voltage impulses of dimension $N_{\tilde{l}}$ and $Q^t = [Q^t_1, Q^t_2, ..., Q^t_{N_{\tilde{F}}}]^T$ is the vector of electric charge flows.  

Note that the continuity equation for static and quasi-static fields is identically satisfied with a charge flow normal to the plane of symmetry and therefore it will not be considered.

### 3.3 Faraday-Neumann law

Considering the primal faces $s_i$, Fig. 3 center, with two boundary edges $l_a, l_b$ normal to the symmetry plane, the Faraday-Neumann law can be written as:  

$$\sum_{i=1}^{4} c_{ij} \mathbf{I}[T_{n+1}, l_j] = \Phi[t_n, s_i] - \Phi[t_{n+1}, s_i]$$  

(5)
where \( c_{ij} \) are the incidence numbers between the inner orientation of the edges \( l_j \) forming the boundary of \( s_i \) and the inner orientation of \( s_i \). Due to the plane symmetry, only the electric voltage impulses along the edges \( l_a, l_b \) contribute in (5). With matrix notation (5) can be rewritten as:

\[
C U[T_{n+1}] = \Phi(t_n) - \Phi(t_{n+1})
\]

being \( C = ||c_{ij}|| \) the incidence matrix, of dimension \( N_s \times N_p \). \( N_p \) is the number of primal nodes equal to the number of primal lines normal to the symmetry plane; \( U = [U_1, U_2, ..., U_{N_p}]^T \) is the vector electric voltage impulses of dimension \( N_p \), relative to the primal edges \( L_p \) normal to the plane of symmetry. Substituting (3) in (6), the Faraday-Neumann law, rewritten in terms of the circulation of the magnetic vector potential, becomes:

\[
U[T_{n+1}] = p(t_n) - p(t_{n+1})
\]

The plane symmetry assures that the electric scalar potential impulse \( V \) at the two points, forming the boundary of \( L_p \), are identical; therefore their difference do not contribute to the electric voltage impulse \( U(T_{n+1}, L_p) \).

### 3.4 Rates of global variables

It is convenient to introduce the temporal rates of global variables associated with a time interval like \( U, F, Q \). If these global variables are approximated to depend linearly on the duration of a sufficiently small interval, then the mean rate approximates the value of the instantaneous rate at the middle instant of the interval, Fig. 2. Therefore the rates of \( U, F, Q \) are the electric voltage, the magnetic voltage and the electric current respectively and are computed as:

\[
U_j(\tilde{t}_n) \approx \frac{U[T_n, \tilde{l}_j]}{T_n}, \quad F_i(t_n) \approx \frac{F[T_n, \tilde{l}_i]}{T_n}, \quad I_j(t_n) \approx \frac{Q[T_n, \tilde{s}_j]}{T_n}
\]

\( U_j(\tilde{t}_n), F_i(t_n) \) and \( I_j(t_n) \) are functions of time instants and their association with \( l_j, \tilde{l}_i \) and \( \tilde{s}_j \) respectively, is accounted for with a subscript index.

Therefore the Ampère’s law (4) and the Faraday-Neumann law (7) can be respectively rewritten as:

\[
\tilde{C} F(t_n) = l(t_n)
\]

\[
U(\tilde{t}_{n+1}) = \frac{p(t_n) - p(t_{n+1})}{T_{n+1}}
\]

in terms of the electric voltage vector \( U = [U_1, U_2, ..., U_{N_p}]^T \), the magnetic voltage vector \( F = [F_1, F_2, ..., F_{N_l}]^T \) and the electric current vector \( l = [l_1, l_2, ..., l_{N_l}]^T \).

### 4 Constitutive equations

The constitutive equations, link the field source variables with the field configuration variables. These equations contain the material parameters and require metric notions such as length and areas. Fields are unavoidably needed to introduce the constitutive laws and the assumption is the uniformity of fields within a triangle or within its subregions.

#### 4.1 Magnetic constitutive equation

The first step in the construction of the magnetic constitutive equation for triangular primal and barycentric dual cells is to work locally element by element and then assemble globally each local material equation. With reference to Fig. 4, an oriented primal cell (a triangle) is considered with
primal nodes \( a, b, c \), primal edges \( l_1, l_2, l_3 \) and primal faces \( s_1, s_2, s_3 \) with inner orientation and local numbering; \( l_1, l_2, l_3 \) are the corresponding portions of dual edges with outer orientation, again with local numbering. It should be noted that the primal faces and the corresponding portions of dual edges are not orthogonal as in the Delaunay-Voronoy tessellation.

The magnetic material is assumed homogeneous inside each primal cell, characterised by the following permeability matrix:

\[
\mu = \begin{bmatrix}
\mu_{1,1} & \mu_{1,2} \\
\mu_{2,1} & \mu_{2,2}
\end{bmatrix}
\]

where the numbers \( \mu_{ij} \) can be constants or functions of the element flux density in the non-linear case. In the following two possible approaches are described to derive the local constitutive matrix between the vector of magnetic voltages \( \vec{F}_e = [F_1, F_2, F_3]^T \) and the flux vector \( \vec{\Phi}_e = [\Phi_1, \Phi_2, \Phi_3]^T \) both related to an element \( e \). A first approach assumes three subregions with uniform fields, \[22\]; it can be extended to quadrilateral primal cells. A second approach assumes the uniformity of fields on the whole triangle; it implies a sensible simplification but it is applicable to simplexes only. Because of Gauss’s law, in the case of triangles, the uniformity in subregions implies the uniformity in the whole triangle but it is not true for quadrilateral elements.

4.1.1 Uniformity subregions

It is convenient to introduce for each element an influence region around each vertex \( a, b, c \) delimited by a couple of portions of dual edges. In each influence region \( a, b, c \) the magnetic field \( H_a, H_b, H_c \) and the flux density \( B_a, B_b, B_c \) vectors, are assumed uniform; each vector has two components \((x, y)\) or \((r, z)\) depending on the 2D problem (plane or axisymmetric).

The three magnetic voltages \( F_1, F_2, F_3 \) along the three portions of dual edges \( \tilde{l}_1, \tilde{l}_2, \tilde{l}_3 \) respectively can be expressed as in Table 2:

<table>
<thead>
<tr>
<th>Region ( a )</th>
<th>Region ( b )</th>
<th>Region ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\begin{bmatrix} F_2 \ F_3 \end{bmatrix} = \begin{bmatrix} \tilde{l}_2 \ \tilde{l}_3 \end{bmatrix} ] ( H_a = \tilde{L}_a H_a )</td>
<td>[\begin{bmatrix} F_1 \ F_3 \end{bmatrix} = \begin{bmatrix} \tilde{l}_1 \ \tilde{l}_3 \end{bmatrix} ] ( H_b = \tilde{L}_b H_b )</td>
<td>[\begin{bmatrix} F_1 \ F_2 \end{bmatrix} = \begin{bmatrix} \tilde{l}_1 \ \tilde{l}_2 \end{bmatrix} ] ( H_c = \tilde{L}_c H_c )</td>
</tr>
</tbody>
</table>

where \( \tilde{l}_1, \tilde{l}_2, \tilde{l}_3 \) are row vectors, of dimension 2, with direction of the portion of dual edge and orientation induced by the right-screw rule from the outer orientation of the dual edge itself. The continuity of the tangential components of the magnetic field is so automatically satisfied.
\( \bar{L}_a, \bar{L}_b, \bar{L}_c \) are 2x2 non-singular matrices that can be inverted and the corresponding inverse can be modified by introducing a column of zeros as follows:

\[
A_a = \begin{bmatrix} 0 & x & x \\ 0 & x & x \end{bmatrix}, \quad A_b = \begin{bmatrix} x & 0 & x \\ x & 0 & x \end{bmatrix}, \quad A_c = \begin{bmatrix} x & x & 0 \\ x & x & 0 \end{bmatrix}
\]

(12)

where \( x \) indicates a non-null element. Therefore the magnetic fields \( H_a, H_b, H_c \) in each influence region, can be deduced from the local vector of magnetic voltages \( F^e \) as:

\[
H_a = A_a F^e, \quad H_b = A_b F^e, \quad H_c = A_c F^e
\]

(13)

Due to the plane symmetry of the field, only the fluxes \( \Phi_1, \Phi_2, \Phi_3 \) relative to the primal lateral faces \( s_1, s_2, s_3 \) need to be considered, Fig. 5. From the flux densities \( B_a, B_b, B_c \) in each region of influence \( a, b, c \) and using the additivity of the flux on portions of the primal faces, the fluxes can be expressed as:

\[
\begin{align*}
\Phi_1 &= \Phi_{1b} + \Phi_{1c} = \frac{1}{2} s_1 B_b + \frac{1}{2} s_1 B_c \\
\Phi_2 &= \Phi_{2a} + \Phi_{2c} = \frac{1}{2} s_2 B_a + \frac{1}{2} s_2 B_c \\
\Phi_3 &= \Phi_{3a} + \Phi_{3b} = \frac{1}{2} s_3 B_a + \frac{1}{2} s_3 B_b 
\end{align*}
\]

(14)

being \( s_1, s_2, s_3 \) the area row vectors, of dimension 2, normal to the primal faces \( s_1, s_2, s_3 \); the orientation of \( s_1, s_2, s_3 \) is congruent with the inner orientation of the corresponding face according to the right-screw rule. The additivity of the fluxes automatically assures the continuity of the normal components of the induction at the primal faces.

Now from the constitutive equation linking the fields:

\[
B_j = \mu H_j, \quad j = a, b, c
\]

(15)

inside each influence region of a primal cell, the local constitutive equation in terms of the global variables can be derived by substituting in (14), the expressions (15) and (13), obtaining:

\[
\Phi_1 = \frac{1}{2} s_1 \mu (A_b + A_c) F^e, \quad \Phi_2 = \frac{1}{2} s_2 \mu (A_a + A_c) F^e, \quad \Phi_3 = \frac{1}{2} s_3 \mu (A_a + A_b) F^e
\]

(16)

From the local flux vector \( \Phi^e \), (16) becomes:

\[
\Phi^e = M^e F^e
\]

(17)

where the 3x3 local matrix \( M^e \) is defined as:

\[
M^e = \frac{1}{2} \begin{bmatrix}
  s_1 \mu (A_b + A_c) \\
  s_2 \mu (A_a + A_c) \\
  s_3 \mu (A_a + A_b)
\end{bmatrix}
\]

(18)
In the numerical formulation for magnetostatic and magneto quasi-static presented here, the inverse \( M_{e,\Phi} \) of the matrix \( M_{e,F} \) need to be considered, such that:

\[
F^{e} = M_{e,\Phi} \Phi^{e} \tag{19}
\]

Note that the local matrix \( M_{e,\Phi} \) is not symmetric, due to the choice of a primal-dual barycentric cell complex; moreover it is non singular and its eigenvalues are all positive. On the other hand, in the case of a Delaunay-Voronoi cell complex the matrix \( M_{e,\Phi} \) becomes diagonal.

### 4.1.2 Uniformity on the whole triangle

From the three fluxes and the area row vectors \( s_1, s_2, s_3 \), the uniform field \( B \) in the whole triangle can be derived as:

\[
B = \begin{bmatrix} s_2 \\ s_3 \end{bmatrix}^{-1} \begin{bmatrix} \Phi_2 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix}^{-1} \begin{bmatrix} \Phi_1 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}^{-1} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \tag{20}
\]

In terms of the local flux vector \( \Phi^{e} \), (20) can be rewritten as:

\[
B = W_{a} \Phi^{e} = W_{b} \Phi^{e} = W_{c} \Phi^{e} \tag{21}
\]

where the \( 2 \times 3 \) matrices \( W_{j} \) with \( j = a, b, c \), correspond to the matrices in (20) with in addition a column of zeros in correspondence of the missing flux. Such a uniform field \( B \) identically satisfies the Gauss law (1). From the inverse of (15) the uniform magnetic field is obtained and then projecting \( H \) along the portion of the dual edge opposite to the node \( a, b, c \) in turn, the vector of the magnetic voltages \( F^{e} \) can be derived as:

\[
F^{e} = \begin{bmatrix} \hat{l}_1 \nu W_{a} \\ \hat{l}_2 \nu W_{b} \\ \hat{l}_3 \nu W_{c} \end{bmatrix} \Phi^{e} = M_{e,\Phi}' \Phi^{e} \tag{22}
\]

where \( \nu \) is the \( 3 \times 3 \) reluctivity matrix and \( M_{e,\Phi}' \) is the non singular non symmetric constitutive matrix having null diagonal elements. \( M_{e,\Phi}' \) has two over three eigenvalues \( \lambda_i > 0, \lambda_j > 0 \) and coincident with those of \( M_{e,\Phi} \) in (19); being the trace\( (M_{e,\Phi}') = 0 \) the third eigenvalue is \( -\lambda_i - \lambda_j \).

A different constitutive matrix can be deduced from the three matrices \( W_{j} \) in (21) by expressing the magnetic field as \( H = \frac{1}{3} \nu (W_{a} + W_{b} + W_{c}) \); therefore projecting it along the potions of dual edges we have:

\[
F^{e} = \frac{1}{3} \begin{bmatrix} \hat{l}_1 \\ \hat{l}_2 \\ \hat{l}_3 \end{bmatrix} \nu (W_{a} + W_{b} + W_{c}) \Phi^{e} = M_{e,\Phi}'' \Phi^{e} \tag{23}
\]

where the constitutive matrix \( M_{e,\Phi}'' \) is now singular with rank 2 but the two non null eigenvalues are coincident with \( \lambda_i, \lambda_j \).

### 4.1.3 Constitutive matrix in terms of \( p \)

The local flux vector \( \Phi^{e} \) can be expressed in terms of the local vector \( p^{e} = [p_1, p_2, p_3]^T \) of the circulations of the magnetic vector potential as in (2), and with matrix notation it becomes:

\[
\Phi^{e} = C^{e} p^{e} \tag{24}
\]

\( C^{e} \) is the local incidence matrix between the inner orientation of the primal edges \( \hat{l}_p \) normal to the plane of symmetry, and the inner orientation of the corresponding primal face \( s_i \). Substituting (24) in (19) or in (22) or in (23) the following local constitutive equation is derived:

\[
F^{e} = M_{e,\Phi} C^{e} p^{e} = M_{e,\Phi}' C^{e} p^{e} = M_{e,\Phi}'' C^{e} p^{e} \tag{25}
\]
where it can be proved that the matrices $M^s \Phi^C \Phi^C = M^s \Phi^C \Phi^C = M^s \Phi^C \Phi^C$ are coincident even though $M^s \Phi$, $M^s \Phi$ and $M^s \Phi$ are not; due to this fact the two approaches for the constitutive equations are equivalent.

In terms of the global vector $p(t_n)$ of the circulations of the magnetic vector potential relative to the primal edges $l_i$, normal to the plane of symmetry, the global constitutive equation can be assembled element by element from (25):

$$F(t_n) = M_\Phi C p(t_n)$$

being $C = \|c_{ij}\|$ the incidence matrix and $M_\Phi C$ the global constitutive matrix of dimension $N_s \times N_t p$.

### 4.2 Ohm’s constitutive equation

In the case of magneto-quasistatic field analysis, the Ohm’s law is required for the conductive regions. For plane (or axisymmetric) problems with electric current normal to the plane of symmetry, the Ohm’s law is relative to a dual face $\tilde{s}_P$ laying on the plane of symmetry, and to the primal line $l_P$ normal to that plane, crossing $\tilde{s}_P$, Fig. 6. The Ohm’s law, written locally for a primal element $e$,

![Diagram](image)

Figure 6: Dual cell $\tilde{s}_P$ (left) and a cluster of primal elements with the common primal edge $l_P$ normal to the plane of symmetry; the generic primal element of the cluster is evidenced on the right with local numbering $(a, b, c)$ of the nodes and of the portions of dual areas internal to it.

with primal nodes $a, b, c$ primal edges $l_a, l_b, l_c$ normal to the plane of symmetry, can be written as:

$$\frac{1}{2} [I_i(t_n) + I_i(t_{n+1})] = \sigma^e \tilde{s}_i \frac{\tilde{s}_i}{l_i} |U_i(\tilde{t}_{n+1}) + U_i^{ext}(\tilde{t}_{n+1})|, \quad \text{with } i = a, b, c$$

(27)

where $\sigma^e$ is the uniform conductivity of the element and $U_i^{ext}(\tilde{t}_{n+1})$ is the external impressed voltage relative to a primal line $l_i$, with $i = a, b, c$; in the case of a passive conductor $U_i^{ext}(\tilde{t}_{n+1}) \equiv 0$.

$I_a, I_b, I_c$ are fractions of the currents, relative to the portions of dual faces $\tilde{s}_a, \tilde{s}_b, \tilde{s}_c$, tailored inside the triangular primal element $e$. Being the dual mesh barycentric, the following identity holds:

$$\tilde{s}_a \equiv \tilde{s}_b \equiv \tilde{s}_c = \frac{s^e}{3}$$

(28)

where $s^e$ is the area of the triangle $e$.

Note that, in (27), the voltages and the currents refer to different temporal grids, Fig. 2, staggered one respect to the other. The assumption that $Q_t$ depends linearly on the dual interval duration,
assures that the current is relative to the mid instant of the dual interval corresponding to the primal instant. Therefore the quantity $\frac{1}{2}[I_i(t_n) + I_i(t_{n+1})]$ is referred to the intermediate dual instant $\tilde{t}_{n+1}$.

Assembling the local equation (27) for each primal element $e$ of the cluster of elements having the common primal edge $I_P$, the global constitutive Ohm’s law can be derived as:

$$I(t_n) + I(t_{n+1}) = 2G[U(\tilde{t}_{n+1}) + U^{ext}(\tilde{t}_{n+1})]$$

(29)

being $G$ the $N_{sc} \times N_{sc}$ diagonal matrix relative to the conductor nodes.

## 5 Solution of the magnetostatic problem

The algebraic system for the solution of the plane magnetostatic field analysis, can be derived by substituting in the Ampère equation (9) the magnetic constitutive equation (26) where the flux vector is expressed in terms of the circulation of the vector potential vector $p$ relative to the edges $I_P$ (individuated by the $N_P$ nodes of the 2D mesh) normal to the plane of symmetry assumed here of unit length. The resulting system of order $N_P$, formally becomes:

$$\tilde{C} M_P C_P = I$$

(30)

where due to the duality of the complexes $\tilde{C} = C^T$; in the implementation the incidence matrix $C$ needs not to be computed or stored and the stiffness matrix in (30) can be easily assembled working element by element. An important property of the stiffness matrix $\tilde{C} M_P C$ can be proved: it is symmetric even though the constitutive matrix $M_P$ is not; $\tilde{C} M_P C$ is sparse having, for each row, a number of non null elements equal to the number of triangles forming a cluster with a node in common.

To solve (30) the boundary conditions have to be considered in addition, assigning $p$ at the boundary nodes. Moreover symmetry conditions, like "flux parallel" or "flux normal" can be easily handled by imposing respectively a null value of the circulations or a leaving unknown the corresponding circulation values relative to the nodes of the symmetry line.

### 5.1 Numerical results in magnetostatic problems

In Fig. 7 a circle of radius $R = 1$ m of relative permeability $\mu_r = 1000$, surrounded by air with an external impressed field $B_0 = 1$ T is considered. Due to the symmetry only one half of the mesh is considered and a linear $p_b$ per unit depth has been imposed on the boundary nodes of the left, top and right boundary. The number of triangles is 4141 and the number of nodes 2108. The analytical value of the field within the circle is $B = 2B_0(\mu_r - 1)/(\mu_r + 2) = 1.9940$ while the computed value averaged among the elements of the circle is $B_c = 1.9519$.

Fig. 8 shows the flux lines produced by an axisymmetric current of 100 kA uniformly distributed in an active conductor, in presence of a non homogeneous media, an axisymmetric square region with relative permeability $\mu_r = 1000$ surrounded by air.

Due to the axisymmetry of the problem, the vector $\psi$ of the circulations of the vector potential for radian has been introduced:

$$\psi = \frac{p}{\alpha}$$

(31)

relative to the primal nodes of the 2D mesh, being $\alpha$ the azimuthal angular width of the cell complexes in $\phi$ direction in a $(r, \phi, z)$ cylindrical reference frame.

The boundary conditions considered here are $\psi_b \equiv 0$, at the boundary primal nodes $P_b$ of the mesh.

In the presence of a non linear media the system (30) has been iteratively solved by means of a quasi-Newton method, the Broyden method [12] with a numerical estimate of the Hessian matrix.
Figure 7: A circular region with $\mu_r = 1000$, surrounded by air is shown; the external uniform field is of 1 $T$.

Figure 8: Flux lines produced by an active axisymmetric conductor current of 100 kA, in the presence of a non homogeneous permeability region with $\mu_r = 1000$. 
kept constant during the iterations. As initial guess of the iteration process, the flux distribution, computed from the homogeneous media, has been considered. The stop criterium considered is the relative discrepancy of the flux distribution in the ferromagnetic region, between two successive iterations. At each iteration the induction field, uniform within each triangle, has been computed from the circulations of the vector potential according to (24) and (21). Fig. 9 shows the resulting flux lines when the non-linear media characteristic, reported in Fig. 10, is assumed for the ferromagnetic region and the active coil is fed with a 100 kA axisymmetric uniform current.

6 Solution of the magneto quasi-static problem

To illustrate the numerical formulation for the solution of a magneto quasi-static field problem, a simple linear axisymmetric Eddy Current Problem (ECP) has been considered: a massive active conductor, of known conductivity is fed with a known external time dependent voltage source in presence of a closed passive massive coil. All the materials are here assumed homogeneous and linear, even though the extension to non homogeneous or non linear media is straightforward. Due to the axisymmetry of the ECP, the resulting eddy currents are all azimuthal (along $\phi$ direction), Fig. 11. With respect to the cell complexes shown in Fig. 11, the vector of the electric voltages per radian can be introduced:

$$u(t_{n+1}) = \frac{U(t_{n+1})}{\alpha}$$

referred to the primal nodes $P$ that are the nodes of the primal 2D mesh, corresponding to the first nodes of the primal edges in the azimuthal direction (e. g. $I_i, I_j, I_k$ in Fig. 11); $\alpha$ is the azimuthal angular width of the cell complexes.

Substituting (32) in the Faraday-Neumann equation (10) the circulation per radian $\psi$ (31), it becomes:

$$u(t_{n+1}) = \frac{\psi(t_n) - \psi(t_{n+1})}{T_{n+1}}$$

6.1 The iterative numerical method

The linear algebraic system to be solved, consists of two sets of equations.

1. Equations relative to the air nodes

In the air primal nodes the homogeneous equation, deduced from (30), holds:

$$\tilde{C} M_\phi C \psi = 0$$

2. Equations relative to the conductor nodes

Substituting in the Ohm’s constitutive equation (29) the vector of current’s given by (30) written for two successive primal time instants $t_n$ and $t_{n+1}$ and the vector of the electric voltages for radian for (33), the following equations can be derived for the conductor nodes:

$$(C^T M_\phi C + \frac{2}{T_{n+1}} G^*) \psi(t_{n+1}) = -(C^T M_\phi C - \frac{2}{T_{n+1}} G^*) \psi(t_n) + 2 G^* u^{ext}(t_{n+1})$$

where $G^* = G/\alpha$ indicates the diagonal matrix of conductances for radian.

Combining the above two sets of equations 1 and 2, the following implicit numerical scheme can be derived:

$$P \psi(t_{n+1}) = Q \psi(t_n) + q(t_{n+1})$$

being $P$ and $Q$, $N_P \times N_P$ matrices whose elements are those of:
Figure 9: Flux lines in presence of a non linear ferromagnetic region surrounded by air; the three snapshots are relative to the initial guess, the second iteration step and the last iteration step respectively.
Figure 10: Permeability curve as a function of the modulus of the element flux density $|B|$.

Figure 11: On the left a detail of the 3D staggered structure of the two cell complexes for axisymmetric field problems; on the right: corresponding 2D view of the primal - dual tessellation with inner outer orientation respectively.
• $C^T M_\Phi C$ for the air nodes;
• $(C^T M_\Phi C + \frac{2}{\tau_{n+1}} G^*)$, $-(C^T M_\Phi C - \frac{2}{\tau_{n+1}} G^*)$ respectively for $P$ and $Q$, for the conductor nodes;

$q(\tilde{t}_{n+1})$ is the time dependent vector whose elements are:
• zeros for the air nodes and the passive conductor nodes;
• $2 G^* u_{ext}(\tilde{t}_{n+1})$ for the active conductor nodes.

If the active conductor has impressed current instead of impressed voltage, then at the nodes of the active conductor the following equation holds:

$$\tilde{C} M_\Phi C \psi = I_{ext}$$

(37)

instead of (35), being $I_{ext}$ the vector of known impressed currents at the dual faces of the active conductor.

The initial conditions are $u_{ext}(\tilde{t}_1) = 0$, $\psi(t_0) = 0$ at the mesh primal nodes; the boundary condition is assumed $\psi_b(t_n) = 0$ for any $t_n$ at the boundary nodes $P_b$ of the primal mesh.

The matrices $P$ and $Q$ are symmetric and the linear system resulting from (36), has been solved by means of the LU factorisation and repeated back substitutions at each time step. It should be noted that the iteration matrix $P^{-1}Q$ has spectral radius less then one, thus assuring the convergence of the scheme (36).

6.2 Numerical results and comparisons

In the numerical implementation of the scheme (36) the components of the vector of the external impressed voltages for radian $u_{ext}(\tilde{t}_n)$ have been set to:

• $u_{ext}^a(\tilde{t}_n) = (1 - \exp(-\frac{\tilde{t}_n}{2\pi}))$ at the primal nodes $P_a$ of the active conductor;
• $u_{ext}^p(\tilde{t}_n) = 0$ at the primal nodes $P_p$ of the passive conductor.

the active and passive conductors have been assumed copper made. The number of primal nodes of the considered mesh is $N_P = 573$ (36 are located on the boundary), while the number of triangular primal elements is $N_V = 1108$. In order to check the accuracy of this method, the standard Finite Element method has been considered, using the ANSYS code, to solve the ECP. Table 3 and Table 4 show the comparison between the current densities calculated according to the Finite Formulation (FF) and the current densities calculated according to the Finite Elements (FE) in five points $A_k$, in the active coil cross-section and in five points $P_k$ in the passive coil cross-section with $k = 1, \ldots, 5$.

Four of these points are located close to the four corners of the passive and of the active conductors respectively, while the fifth point, $P_5$ or $A_5$, is located at the cross-section center of the passive and of the active conductor respectively. Moreover three time instants (2 s, 5 s, 10 s) have been considered.

To perform the comparison in terms of the current densities, an interpolation was necessary because the FE attributes the current densities to the element barycenter while in the FF the current density $J_j = I_j/\hat{s}_j$ is relative to the primal node $P$ internal to the dual face $\hat{s}_j$.

7 Conclusions

The work has illustrated a numerical formulation for static and quasi-static problems deduced from the Finite Formulation; some applications to the solution of plane field problems have also been shown. The examples analysed, show that the finite formulation can be easily applied also in the case of different materials, anisotropic, non homogeneous or non linear; moreover the material
Figure 12: Snapshots of the flux lines diffusion in the active and passive conductors at 2 s, 5 s and 10 s respectively.
Table 3: Current densities $10^5 A/m^2$ in the 5 points $A_k$ of the active conductor.

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 $FE$</td>
<td>7.436</td>
<td>6.405</td>
<td>5.766</td>
<td>5.571</td>
<td>3.284</td>
</tr>
<tr>
<td></td>
<td>7.437</td>
<td>6.538</td>
<td>5.746</td>
<td>5.575</td>
<td>3.279</td>
</tr>
<tr>
<td>5 $FE$</td>
<td>11.80</td>
<td>11.18</td>
<td>9.966</td>
<td>9.883</td>
<td>8.647</td>
</tr>
<tr>
<td></td>
<td>11.99</td>
<td>11.48</td>
<td>10.15</td>
<td>10.09</td>
<td>8.909</td>
</tr>
<tr>
<td>10 $FE$</td>
<td>16.60</td>
<td>15.94</td>
<td>14.10</td>
<td>14.11</td>
<td>13.95</td>
</tr>
</tbody>
</table>

Table 4: Current densities $10^5 A/m^2$ in the 5 points $P_k$ of the passive conductor.

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 $FE$</td>
<td>7.436</td>
<td>1.160</td>
<td>1.685</td>
<td>1.808</td>
<td>1.101</td>
</tr>
<tr>
<td></td>
<td>7.437</td>
<td>1.194</td>
<td>1.754</td>
<td>1.834</td>
<td>1.131</td>
</tr>
<tr>
<td>5 $FE$</td>
<td>2.012</td>
<td>2.508</td>
<td>2.472</td>
<td>3.354</td>
<td>2.311</td>
</tr>
<tr>
<td></td>
<td>2.136</td>
<td>2.634</td>
<td>2.572</td>
<td>3.443</td>
<td>2.455</td>
</tr>
<tr>
<td>10 $FE$</td>
<td>2.162</td>
<td>2.495</td>
<td>2.378</td>
<td>2.862</td>
<td>2.544</td>
</tr>
<tr>
<td></td>
<td>2.269</td>
<td>2.584</td>
<td>2.455</td>
<td>2.903</td>
<td>2.675</td>
</tr>
</tbody>
</table>

proprieties can be different cell to cell. The sources can be easily modelled both in the case of discontinuous and in the case of concentrated sources.

The results obtained in the case of an axisymmetric eddy currents problem proved to be in a good agreement with those from the classical Finite Elements method, based on the differential approach. The implemented finite formulation approach can be extended to the 3D magnetic field analyses.

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References


