Numerical algorithms for the FDiTD and FDFD simulation of slowly varying electromagnetic fields†

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SUMMARY

The simulation of slowly varying electromagnetic fields is possible for very large, realistic problems with finite-difference implicit time-domain (FDiTD) and frequency-domain (FDFD) formulations on the basis of the consistent Finite-Integration Technique (FIT). Magneto-quasistatic time-domain formulations combined with implicit time marching schemes require the repeated solution of real-valued symmetric systems. The solution of driven frequency domain problems usually consists in the solution of one non-Hermitean system. Preconditioned conjugate gradient-type methods are well-suited for this task. They allow the efficient solution even for consistent singular or near-singular systems, which typically arise from formulations for slowly varying electromagnetic fields using the Maxwell-Grid-Equations of the FI-Method. Numerical results for TEAM workshop 11 benchmark problem and for a large practical problem, a shading ring sensor, show that the presented algorithms are capable of solving realistic problems for large numbers of unknowns in acceptable calculation times on contemporary medium sized workstations. Copyright © 1999 John Wiley & Sons, Ltd.

1. Introduction

The explicit Finite-Difference Time-Domain Leap-Frog algorithm (FDTD) developed by Yee in 1966† is today a well-developed and thoroughly analysed numerical tool for the efficient and fast calculation of electromagnetic fields in the high-frequency domain, where wave propagation phenomena have to be included and where the wave lengths are small or of the same size as the dimension of the calculation domain. The FDTD method can elegantly be described and analysed using the Finite-Integration Technique (FIT) developed by Weiland in 1977, in which Maxwell’s equations are consistently discretized into a set of matrix equations, the so-called Maxwell-Grid-Equations.

In the case of slowly variable electromagnetics fields, however, where the dominant wavelengths of the problem are large when compared to the dimensions of the device under consideration, the number of required time steps for the solution of electromagnetic fields with the explicit FDTD algorithm may become unsuitable due to the Courant–Friedrich–Levy-stability limit for the timestep length. In these cases the Maxwell-Grid-Equations pose a system of ordinary differential equations which are stiff. In order to still be able to use the explicit method for slowly varying problems the so-called reduced-ε₀-approach was proposed by Holland, where the stiffness of the system is reduced with a non-physical increase of the vacuum permittivity ε₀.

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For this a scaling factor $e := x \cdot e_0$, $x \geq 1$, is introduced resulting in larger stable timesteps $\Delta t_{\text{new}} := \sqrt{x} \Delta t_{\text{CFL}}$ for the Leap-Frog-scheme. This approach was used successfully in Reference 6, where it allowed the calculation for the quasistatic magnetic fluxes and eddy currents with acceptable errors. However, the proper choice of the scaling factor $x$ is difficult and usually requires tests on the expected error of the solutions.

Other approaches use the explicit FDTD method to calculate the effects of extremely low-frequency fields of high voltage transmission power lines (50 Hz/60 Hz) on human tissue. In Reference 7, a high-frequency solution is performed and the low-frequency fields in the body tissue are obtained linearly scaling down the high-frequency solution. A different approach is described in Reference 8 for these low frequencies using two time-ramped incident plane waves from opposite directions to allow the decoupled calculation of electric and magnetic induction in human bodies. These two approaches do not consider non-linear field dependencies from materials that may eventually have to be taken into account.

For the solution of general stiff equations in the mathematical literature usually implicit time marching schemes are considered, where each timestep is combined with the solution of one or several (non)-linear systems of equations. For this approach a wide number of magneto-quasistatic formulations in combination with finite-element-discretization schemes already exist. The most recent developments in this area for the transient Maxwell-Grid-Equations are presented in this paper as the so-called Finite-Difference-implicit-Time-Domain methods (FDiTD). After a short presentation of the basic ideas of the FIT the time domain formulations for magnetoquasistatic problems and suitable implicit time integration methods are presented. For time-harmonic steady-state problems with linear material properties it is possible to transform the Maxwell-Grid-Equations into the frequency domain. The numerical treatment of a general Helmholtz-type curl–curl-equation for low excitation frequencies faces difficulties arising from the near-singularity of the curl–curl part similar to the time-domain formulations. For electro-quasistatic fields, where no eddy currents have to be considered, a different, specialized formulation may be applied. For the solution of the resultant large, linear, symmetric, complex-valued systems, modern Krylov subspace iteration methods may be used.

2. The Finite Integration Technique

FIT\textsuperscript{2} is a proven consistent discretization scheme which maps Maxwell's equations in the integral form in terms of tensions

$$E_i := \int_{L_i \subset G} \mathbf{E} \cdot \mathbf{s} \, \operatorname{ds}, \quad H_i := \int_{\mathcal{L}_i \subset \mathcal{G}} \mathbf{H} \cdot \mathbf{s} \, \operatorname{ds}$$

(1)

along gridcell edges $L_i$, $\mathcal{L}_i$ and fluxes

$$B_i := \int_{A_i \subset G} \mathbf{B} \cdot \mathbf{d} \mathbf{A}, \quad D_i := \int_{\mathcal{A}_i \subset \mathcal{G}} \mathbf{D} \cdot \mathbf{d} \mathbf{A}$$

(2)

through grid facets $A_i$, $\mathcal{A}_i$ onto a dual gridcell complex \{G, $\mathcal{G}$\} (Figure 1). The single degrees of freedom on the grid edges and facets are summarized into the grid voltage vectors $\mathbf{e} := (E_i)$, $\mathbf{h} := (H_i)$ and flux vectors $\mathbf{b} := (B_i)$, $\mathbf{d} := (D_i)$.

The grid adjacency relations of the tension and flux components within Maxwell's equations yield matrices $\mathbf{C}$, $\mathbf{\tilde{C}}$ and $\mathbf{S}$, $\mathbf{\tilde{S}}$. Thus an exact representation of Maxwell's equations is possible.
within the following Maxwell-Grid-Equations (MGE) using integrated fields such as the facet fluxes in the vectors $b, d$ and the grid tensions in the vectors $e, h$ instead of the field components:

$$Ce = -\frac{d}{dt} b \quad (3)$$

$$\tilde{C}h = \frac{d}{dt} d + j \quad (4)$$

$$Sb = 0 \quad (5)$$

$$\tilde{S}d = q \quad (6)$$

Here $j$ denotes the current vector and $q$ is a vector of space charges. In this scheme an approximation only occurs in the material equations between related tensions and fluxes by $d = D, e, j = D, e + j, \text{and } D, b = h$. The vector $j$ corresponds to an externally applied current. The basic properties of the fields are maintained when moving from $\mathbb{R}^3$ to the gridspace $\{G, \tilde{G}\}$, and matrix theory can be used to analyse the properties of fields without having to solve the equations numerically. The matrices $C$ and $\tilde{C}$ correspond to the curl-operators on the primary and on the dual grid and contain only information on the grid topology having only integer entries $\{-1, 0, 1\}$. The same holds true for the discrete divergence (source)-operators $S$ and $\tilde{S}$. The predefined duality of the gridcell complexes $\{G, \tilde{G}\}$ results in the equation

$$C = \tilde{C}^T \quad (7)$$

whereas from topology the matrix equation

$$SC = \tilde{S}C = 0 \quad (8)$$

can be derived. These two key properties to the Maxwell-Grid-equations are also known to hold for finite-element-formulations using lowest order Nédélec edge elements.\textsuperscript{10} This yields the
correspondence to the analytical relations

\[ \mathbf{C} \mathbf{S}^{T} = 0 \leftrightarrow \text{curl grad} \equiv 0 \]  
(9)

\[ \mathbf{S} \mathbf{C} = 0 \leftrightarrow \text{div curl} \equiv 0 \]  
(10)

and the essential to the stability and to the efficiency of the numerical schemes used for time integration.\(^{5,11}\)

Non-orthogonal grids\(^{12}\) and consistent schemes for a local \(h\)-refinement (so-called subgridding schemes)\(^{13}\) were introduced to the FI-method to circumvent errors of conventional staircase approximations for structures with curved surfaces. The latest development for a reduction of errors arising from the geometry approximation includes the so-called perfect boundary approximation of curved grid boundaries to perfectly conducting materials.\(^{14}\)

### 3. Time domain formulations

#### 3.1. Transient magneto-quasistatic formulations for the FI-method

For the FI-Method a \(b\)-oriented transient formulation is favoured which starts from the integral formulation for a modified magnetic vector potential \(A^{*}\), whose path integrals \(A_{i} = \int_{L_{i}} G A^{*} \cdot d\mathbf{s}\) along the grid edges \(L_{i}\) of the primary cell-complex \(G\) are summarized in a vector \(\mathbf{a}\) for which

\[ \mathbf{e} = -\frac{d}{dt} \mathbf{a}, \quad \mathbf{C} \mathbf{a} = \mathbf{b} \]  
(11)

holds.

Using this primary variable, a magneto-quasistatic formulation for slowly varying transient fields can be formulated as a first-order system

\[ \begin{align*}
\mathbf{C} \mathbf{D}_{x} \mathbf{C} \mathbf{a}(t) + \mathbf{D}_{x} \frac{d}{dt} \mathbf{a}(t) &= \mathbf{j}_{c}(t) \\
\end{align*} \]  
(12)

subject to a consistent initial solution \(\mathbf{a}_{0} := \mathbf{a}(t_{0})\) and a consistent current vector \(\mathbf{j}_{c}\) for which \(\mathbf{S} \mathbf{j}_{c}(t) = 0\) has to hold.

In the non-conducting regions of the computational area, the diagonal matrix of conductivities \(\mathbf{D}_{x}\) will be singular and thus a differential–algebraic system of equations (DAE) of index 1 has to be solved. In case of no conductive material in the computational domain, equation (12) reduces to a magnetostatic vectorpotential formulation

\[ \mathbf{C} \mathbf{D}_{x} \mathbf{C} \mathbf{a} = \mathbf{j}_{c} \]  
(13)

#### 3.2. Gauging techniques

The formulations (12) and (13) initially do not yield unique solutions \(\mathbf{a}(t)\). In the conducting regions the implicit gauging

\[ \mathbf{S} \mathbf{D}_{x} \frac{d}{dt} \mathbf{a}(t) = 0 \]  
(14)
holds for the eddy currents. This constraint on the time derivative of \( \mathbf{a} \) requires that a consistent initial solution \( \mathbf{a}_0 \) is given. In the non-conducting regions irrotational solution parts belonging to the nullspace of the curl-operator \( \mathbf{C} \) may occur, which initially are not considered in equation (12).

This requires the introduction of an additional gauging for the grid allocated vector potential, which is the equivalent of trying to restrict the vectorspace of possible grid solutions \( \mathbf{a} \) such that the curl operator will describe a one-to-one correspondence between \( \mathbf{b} \) and \( \mathbf{a} \) in the non-conducting regions.

If the metallic boundaries of the computational region are simply connected, irrotational fields can only be caused by nodal charges \( \mathbf{q} \) in the non-conducting regions operating as discrete sources to these fields.\(^{15}\) The components of the vector potential are allocated on the edges of the grid, these nodal charges of the primary grid \( G \) however require to consider the facet fluxes of the dual grid volumes enclosing these charges in order to apply the discrete source matrix \( \mathbf{S} \). For this we define \( \mathbf{D} := 1/\varepsilon_0 \mathbf{D}_\varepsilon \) and now grid voltages \( \mathbf{v} \) with \( \mathbf{Cv} = 0 \) and \( \mathbf{S} \mathbf{Dv} = \mathbf{q} \) have a unique representation as a gradient of a nodal potential

\[
\mathbf{v} = \mathbf{S}^T \Phi
\]

and it holds \( \mathbf{S} \mathbf{D} \mathbf{S}^T \Phi = \mathbf{S} \mathbf{Dv} = \mathbf{q} \). Thus a Coloumb gauge for equations (12) and (13) can be rewritten as \( \mathbf{SDa} = 0 \).

The diagonal norm matrix \( \mathbf{D}_N \) is set up to have a non-zero entry pattern such that the \( \text{grad-div} \)-matrix enforces the Coulomb gauge only in the non-conducting regions. Thus equation (12) can be rewritten as

\[
[\mathbf{C} \mathbf{D}_C + \mathbf{D} \mathbf{S}^T \mathbf{D}_N \mathbf{S} \mathbf{D}] \mathbf{a}(t) + \mathbf{D}_N \frac{d}{dt} \mathbf{a}(t) = \mathbf{j}_v(t)
\]

In the case of multiply connected metallic boundaries within the computational region this regularization will not be sufficient. In this case grid vectors \( \mathbf{v} \) depending on the topology will also have to be considered for which at the same time \( \mathbf{Cv} = 0 \) and \( \mathbf{S} \mathbf{Dv} = 0 \) holds.\(^{17}\) The irrotational solution parts however do not hinder a unique solution for the vector of the magnetic facet fluxes with \( \mathbf{b} = \mathbf{Ca} \).

Another possible gauging of the grid solutions \( \mathbf{a} \) in the non-conducting regions arises from a graph theoretical point of view. For the representation of a flux \( \mathbf{B}_j \) through a grid facet \( F_j \) the \( j \)th row of the integer values curl-matrix \( \mathbf{C} \) sums up the path integrals \( \mathbf{A}_j \) along the boundary edges \( L_j \cap \partial F_j \neq 0 \), where the signs of summations depend on the prescribed orientations of the facet boundary and the cell edge. However, for a consistent representation it is sufficient if the one degree of freedom of the flux is merely coupled to one degree of freedom represented with a path integral along a chosen facet boundary edge, whereas all the other degrees of freedom may be set to zero.

Reconsidering the equation \( \mathbf{b} = \mathbf{Ca} \) on a gridcell complex with \( N_{\text{Edges}} \) gridcell edges and grid nodes \( N_{\text{Nodes}} \) one may now define a connected tree-graph of \( N_{\text{Nodes}} - 1 \) gridcell edges, which also yields the cotree-graph of the remaining \( N_{\text{Edges}} - N_{\text{Nodes}} + 1 \) edges. The degrees of freedom of a grid voltage vector \( \mathbf{v} \) corresponding to the tree-graph edges of \( G \) are set to zero and the remaining components allocated on the cotree-graph grid edges will suffice to describe a solenoidal grid facet flux vector \( \mathbf{b} \) with the relation \( \mathbf{b} = \mathbf{Cv} \), which now describes a one-to-one correspondence. On a Cartesian grid about two third of all degrees of freedom will be set to zero in this approach.
This so-called *cotree-gauging* was developed by Rubinacci and Albanese\(^\text{18}\) for Finite-Element-formulations using Nédélec edge elements. This corresponds to the view that any solution \(\mathbf{a}\) of equation (12) has to be \(\mathbf{D}\)-orthogonal in the non-conducting regions to an irrotational vector \(\mathbf{v} = \mathbf{S^T}\Phi\) which yields a weakly imposed Coloumb gauge

\[
0 = \mathbf{a}^T\mathbf{Dv} = \mathbf{a}^T\mathbf{D}^T\mathbf{S^T}\Phi = \Phi^T(\mathbf{S}\mathbf{D}\mathbf{a}) = 0
\]  

(17)

for the vectors \(\mathbf{a}\) and \(\mathbf{v}\) restricted to the non-conducting regions.

Biro *et al.*\(^\text{19}\) and Kameari\(^\text{20}\) showed that this approach has also serious drawbacks. The choice of the tree- and cotree-graphs corresponding to the choice of a irrotational vector \(\mathbf{v}\) in the grid complex \(\mathcal{G}\) is not uniquely determined. The omission of certain adjacency relations within the discrete problem usually will result in badly conditioned linear algebraic systems of equations for which large numbers of preconditioned conjugate gradient (PCG) iteration steps have to be performed even for small problem dimensions. So far only heuristical knowledge has been gathered on the optimal choice of the tree-edges, which is usually performed to reduce the number of strongly coupled facets and thus favours tree-edges chosen in spatial directions, along which the facet fluxes show the least variation.

The cotree-gauging approach however does allow to use available direct and iterative blackbox methods for the solution of the linear systems arising from the numerical schemes. Such blackbox methods usually contain (incomplete) matrix factorization processes which usually do not cope well with any non-regularity within the discrete formulations.

If the grad–div- and the cotree-gauging approaches for the Coulomb gauges \(\mathbf{S}\mathbf{D}\mathbf{a} = 0\) are extended to the whole computational grid, a scalar potential vector \(\Phi\) can be introduced to the grid nodes in the conductive media in order to model jumps of the electric field components normal to interfaces of different conductivity. Thus the continuity equation (14) has to be used to couple the grid vectors of vector potential path integrals \(\mathbf{a}\) and the electric scalar nodal potential \(\Phi\) in the conducting regions and in addition the magneto-quasistatic formulation (12) can be rewritten as

\[
\mathbf{C}\mathbf{D}_s\mathbf{C}\mathbf{a}(t) + \mathbf{D}_s \frac{d}{dt} \mathbf{a}(t) + \mathbf{D}_s\mathbf{S}^T\Phi(t) = \mathbf{j}_s(t)
\]

\[
\mathbf{S}\mathbf{D}_s \left[ \frac{d}{dt} \mathbf{a}(t) + \mathbf{S}^T\Phi(t) \right] = 0
\]

(18)

for initial values \(\mathbf{a}_0 := \mathbf{a}(t_0), \Phi_0 := \Phi_0\). This approach will require a more complicated data structure in the numerical schemes than equation (12). The formulation will not have higher storage requirements than formulation (12) only in connection with the cotree-gauging and a data structure that will actually take advantage of the reduced number of degrees of freedom of \(\mathbf{a}\) within the conductive regions. Thus for practical calculations the approach (18) is not considered in this paper.

A third and most practical approach to the gauging problem does not consider any gauge within the problem formulation, but relies on the numerical methods to cope with the resultant singular systems. Since we are only interested in the rotational parts of solutions \(\mathbf{a}\) in the non-conducting regions of the initial formulation (12), any irrotational part to such a solution may be acceptable if it does not negatively affect the numerical solution process. This, however, is a property of CG-type solution methods which can be applied to the resulting linear systems without suffering from ill-conditioning as one might expect.
3.3. Implicit time integration methods

Formulations (12) and (16) pose a system of differential-algebraic equations of differentiation index 1 (DAE). The differentiation index describes, how many times the algebraic variables, i.e. the components of \( a \) belonging to grid edges in the non-conducting regions, have to be differentiated in time to receive a system of ordinary differential equations.\(^{21}\) A time integration scheme is defined to be of order \( p \), if its global error, i.e. the difference \( v(t_n) - v^* \) between the exact solution \( v(t_n) \) at time \( t_n = t_0 + n\Delta t \) of the problem after \( n \) timesteps and the solution of the timestepping scheme \( v_n \) behaves asymptotically like \( O(\Delta t^p) \), when the length of the time steps \( \Delta t \to 0 \). It is known that for stiff ordinary differential equations (ODE) and differential-algebraic equations (DAEs) of index 1 the order of the method may differ from the order observed in the methods for non-stiff systems.\(^{4,9}\)

The \( b \)-oriented formulations given by (12) and (16) can be written in the form

\[
D_\kappa(v, t) \frac{d}{dt} v(t) + K(v, t) v(t) = j_\kappa(t), \quad v(t_0) = v_0 \tag{19}
\]

where a difference has to be made between the following cases:

- **\( D_\kappa \)** is regular and independent of \( v \) and \( t \). In this case we have a stationary conductivity in the whole computational domain and we have to consider a system of ordinary differential equations for which initially the non-stiff order conditions will hold. If the regularity of \( D_\kappa \) arises from an artificial regularization in which even for the non-conductive regions a small conductivity is assumed, the problem may become stiff again and we have to expect the solution methods to converge as if applied in the DAE index-1 case.

- **\( D_\kappa \)** is singular and independent of \( v \) and \( t \). In this typical case we also consider regions in the computational domain without conductive materials and the system of equations (19) now denotes a differential-algebraic system of linear equations (DAE) with index 1. If also the matrix \( K := K(v, t) \) is constant, we have the case of a constant coefficient system, for which the standard implicit timestepping methods may be applied. This situation occurs in the case of metallic materials with a linear constitutive relation between \( h \) and \( b \). The case of non-linear, ferromagnetic material behaviour will lead to the case of a quasilinear DAE, which are analysed with respect to uniqueness and existence in Reference 22.

- **\( D_\kappa \)** is singular and depends on \( v \) or \( t \). Such a case must be considered if temperature-dependent conductivities or moving conductors exist in the computational domain. Such problems are covered by the theory of fully implicit non-linear DAE index-1 problems.

In Reference 23 these different cases have been first distinguished for magneto-quasistatic DAE systems.

For Finite-Element-formulations usually one-step timestepping methods such as the implicit \( \theta \)-algorithms\(^{51}\) are applied. For DAE index 1 systems (19) of the FI-method with constant, but possibly singular conductivity matrices \( D_\kappa \) these methods yield (non-)linear systems

\[
\left[ \frac{1}{\theta \Delta t} D_\kappa + K(v) \right] v_{n+1} \left[ \frac{1}{\theta \Delta t} D_\kappa + \frac{1 - \theta}{\theta} K(v) \right] v_n + j_{\kappa,n+1} + \left( \frac{1 - \theta}{\theta} \right) j_{\kappa,n} \tag{20}
\]

which must be solved for each new discrete time solution \( v_{n+1} \). For \( \theta = 1 \) we consider the fully implicit first order backward differentiation scheme (BDF1) which is of first order and L-stable,
i.e. which numerically damps errors with small time constant even for large timesteps. The value \( \theta = 2/3 \) corresponds to the so-called Galerkin-method which is of first order. The value \( \theta = 1/2 \) yields the trapezoidal rule, which is \( O(\Delta t^2) \) for all the above problems, but which will not damp out high-frequency error components in the solution and is thus prone to error accumulation\(^4\) and also not well-suited for problems with abrupt changes in the driving term \( \mathbf{j}_h(t) \). Both methods are unconditionally stable (A-stable). An important property of the transient formulation (12) arises from the fact that for all time discrete solutions \( \mathbf{a}_{n+1} \) of the formulation (20) the equality

\[
\tilde{\mathbf{S}} \mathbf{D}_n \mathbf{a}_{n+1} = \tilde{\mathbf{S}} \mathbf{D}_n \mathbf{a}_n \quad \text{for} \quad n \geq 0
\]  

holds, which thus enforces the magneto-quasistatic continuity equation (14) in a first-order approximation.

Applicable higher order one-step schemes arise from the theory of implicit Runge–Kutta methods (IRK), which can be written for system (19) as

\[
\mathbf{D}_n \mathbf{v}_i = - \mathbf{K} \left( \mathbf{v}_n + \Delta t \sum_{j=1}^{s} a_{ij} \mathbf{v}_j \right) \left( \mathbf{v}_n + \Delta t \sum_{j=1}^{s} a_{ij} \mathbf{v}_j \right) - \mathbf{j}_h(t_n + c_i \Delta t) \quad i = 1, \ldots, s
\]

\[
\mathbf{v}_{n+1} := \mathbf{v}_n + \Delta t \sum_{j=1}^{s} b_j \mathbf{v}_j
\]

where the stage derivatives \( \mathbf{v}_i \) are approximations of \( \frac{d}{dt} \mathbf{v}(t_n + c_i \Delta t) \). The coefficients which define the implicit Runge–Kutta method are described using so-called Butcher-diagrams

\[
\begin{array}{c|cccc}
   c_i & a_{11} & a_{12} & \cdots & a_{1s} \\
   c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
   \vdots & \vdots & \vdots & \ddots & \vdots \\
   c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
   \hline
   b_1 & b_2 & \cdots & b_s
\end{array}
\]

where \( c = (c_1, \ldots, c_s)^T \) yields the times \( t_n + c_i \Delta t \) for the evaluation of the interior stages of the method, \( \mathbf{A} = (a_{ij}) \) is the coefficient matrix of the method and \( \mathbf{b} = (b_1, \ldots, b_s)^T \) is the vector of weights for the intermediate solutions. A nice feature of IRK-methods with \( c_s = 1 \) when applied to equations (12) or (16) is that we get the approximation of vector \( \frac{d}{dt} \mathbf{a}_{n+1} \) for the calculation of the eddy currents \( \mathbf{j}_{h,n+1} := -\mathbf{D}_n \frac{d}{dt} \mathbf{a}_{n+1} \) without having to evaluate an additional difference quotient.

In the implicit singly diagonal Runge–Kutta \( s \)-stage methods (SDIRK\((s)\)) only regular, lower diagonal matrices \( \mathbf{A} \) with a single eigenvalue \( \lambda \) are considered. These methods require \( s \) linear systems to be solved for each new time step\(^9\) instead of a non-linear system of dimension \( s \cdot n \) in the general IRK-methods, where \( n \) is the dimension of the vector \( \mathbf{v} \). This generally allows the reuse of a once performed (incomplete) \( LU \)-factorization of the matrix \( \frac{1}{\Delta t} \mathbf{D}_n + \mathbf{K} \) for the direct solution of the linear system (or the preconditioner of an iterative solution method) and also keeps memory requirements small. The usual number of degrees of freedom in the Maxwell-Grid-Equations will require the iterative solution of the systems which in practice limits the applicability to methods with \( s \leq 2 \). Especially recommended for DAEs of index 1 are

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IRK-methods for which the last row of the non-singular matrix $A$ in the Butcher-diagram coincides with the vector $b$ as

$$a_{si} = b_i, \quad i = 1, \ldots, s$$

(25)

Such methods are called **stiffly accurate** and only for such IRK-methods the global error for both the algebraic and the differential components of the DAE will be of order $p$. Stiff accuracy of an IRK-method implies L-stability. For methods which are not stiffly accurate we may get orders $r \leq p$ for the global error in the algebraic components of $a$ in the non-conducting regions. Thus we have to deal with order reduction for the DAEs when compared to the ODE case [4, Theorem 1.1, p. 380]. Higher order $SDIRK$-methods have first been considered in Reference 25 for transient magneto-quasistatic formulations, but the presented methods were not stiffly accurate.

One-stage $SDIRK1$ methods with

$$\begin{array}{c|c}
\lambda & \lambda \\
\hline
1 & 1 - \lambda & \lambda \\
\end{array}$$

are almost identical to the $\theta$-algorithms apart from the evaluation of the driving term. Here the $BDF1$-method ($\lambda = 1$) is stiffly accurate, but it has lowest possible order anyway. For $s = 2$ an $L$-stable, stiffly accurate $SDIRK2$ method of second order is available\textsuperscript{26} with

$$\begin{array}{c|c|c}
\lambda & \lambda & \lambda \\
\hline
1 & 1 - \lambda & \lambda \\
1 - \lambda & \lambda & \lambda \\
\end{array}$$

for $\lambda = 1 - \sqrt{2}/2$.

Other suitable time integration methods are available with the implicit $k$-step backward-differentiation-formulas ($BDF(k)$) of Gear which were among the earliest methods considered for index-1 DAEs. There exist unconditionally stable $BDF$-formulations up to order 2 and so-called $A(\infty)$-stable $BDF$-methods up to order 6, where the stability region no longer is the whole negative complex half plane. It still contains the real negative axis, which is sufficient due to the symmetry of the system matrices arising from formulations (12) and (16). Since the eigenvalues of the symmetric system matrix $-K(v,t)$ are all real-valued and non-positive, these higher-order methods may also be applied.

For practical purposes the second-order multistep backward differentiation formula ($BDF2$) of Gear\textsuperscript{9} can be applied with

$$\begin{bmatrix}
\frac{3}{2\Delta t} D_k + K(a_{n+1})
\end{bmatrix} a_{n+1} = \frac{2}{\Delta t} D_k a_n - \frac{1}{2\Delta t} D_k a_{n-1} + j_{e,n+1}$$

(26)

A start-up problem arises for the solution at the first timestep $a_1$ within the $BDF2$-method, since $a_{-1}$ is not available. This problem can be solved doing the first step with the second order, stiffly accurate $SDIRK2$ method.\textsuperscript{26}

Although these multistep methods are the basis for general-purpose time integration software packages for stiff ODE and DAE of index 1 systems like $DASSL$\textsuperscript{9} or $VODE$,\textsuperscript{27} they do not seem to be commonly used in the transient numerical eddy current calculation. This may be due to the fact that the implementation of variable timestep schemes becomes more difficult than in the equidistant case\textsuperscript{27} and that multistep methods must be restarted after every discontinuity.\textsuperscript{9}
In case of symmetric formulations as the transient first-order curl–curl-equations (12) or (16) arising from the FI-method, they allow for high order schemes with only one (non)-linear system to be solved per timestep whereas in the SDIRK-s-stage one-step methods s systems have to be solved per timestep.

3.4. Iterative solution of the time-discretized formulations

The linear system of equations which have to solved for the time discretized formulations are all of the type

\[
\frac{1}{\lambda \Delta t} D_k + K \mathbf{v}_{n+1} = \mathbf{j}_{\text{RHS}}
\]  

(27)

where the right-hand side vector \(\mathbf{j}_{\text{RHS}}\) consists of a sum of vectors such that \(\mathbf{S} D_k \mathbf{v}_{n+1} = \lambda \Delta t \mathbf{S} \mathbf{j}_{\text{RHS}}\) enforces equation (14). The solution of these symmetric, positive-(semi)-definite linear algebraic systems with typical order of \(O(10^4)\) to \(O(10^7)\) unknowns is performed with the preconditioned conjugate-gradient method (PCG).\(^{28}\)

So far a symmetric-successive-overrelaxation-preconditioner (SSOR) has been recognized as the most robust preconditioning method even for the non-gauged formulation, which in addition does not need an initial matrix decomposition. Other methods such as the (modified) incomplete Cholesky ((M)IC(0))-implementation\(^{28}\) often showed breakdowns in the set-up phase due to the possible singularity of the linear systems and since the matrices under consideration have no M-matrix property.\(^{29}\) An asymptotically better preconditioning method may be expected from the multigrid approach which was especially developed for the non-gauged curl–curl-equations of edge-element-based finite-element-formulations.\(^{30}\)

An important result for the non-gauged magneto-(quasi-)static formulations (12) and (13) arises from Reference 31 for the so-called Krylov subspace iteration methods, in which the approximate solution \(v_k\) of a linear quadratic algebraic system \(Ax = b\) is iteratively calculated with the construction of a basis \(V_k := [v_1, \ldots, v_k]\) for a Krylov subspace such that for an initial guess \(x_0\)

\[
x_k = x_0 + V_k \mathbf{y}_k
\]  

(28)

holds, where \(\mathbf{y}_k\) is the coefficient vector for the linear combination of the constructed vectors. The conjugate-gradient-method also belongs to this type of methods. It is shown that for a singular system as e.g. given with the magnetostatic equation (13), the iterated approximations \(v_k\) converge to a correspondent solution

\[
v = (\tilde{C} D, C)^* j_e + P_{N(C\tilde{D}, C)} Rg(C\tilde{D}, C) v_0
\]  

(29)

where the matrix \((\tilde{C} D, C)^*\) is a pseudo-inverse of the curl–curl matrix, \(v_0\) is a start vector of the iteration and \(P_{N(C\tilde{D}, C)} Rg(C\tilde{D}, C)\) is an oblique projector which has a range consisting of the nullspace \(N(\tilde{C} D, C)\) of the curl–curl matrix and a nullspace consisting of the range \(Rg(\tilde{C} D, C)\) of the curl–curl matrix corresponding to all solenoidal field vectors with \(C a \neq 0\). The whole iteration process only takes place in the vectorspace spanned by the eigenvectors corresponding to the non-vanishing, positive eigenvalues of the system matrix, and any part of the initial vector \(x_0\) not belonging to this space is kept in the final solution.

A necessary prerequisite for this result is the consistency of the current excitation vector, i.e. \(j_e \in \text{Range}(\tilde{C} D, C)\) must hold. This explains the observation that PCG-methods converge for
linear systems arising from consistently discretized (quasi)static, singular formulations as if these were gauged and that any irrotational field part of the initial approximation \( v_0 \) is kept within the final solution. This property is proposed as weak gauging property of the Krylov subspace methods for consistent discretizations and will also hold for complex-valued frequency domain formulations presented in the following.

4. Frequency domain formulations

The earliest approaches to calculate slowly varying electromagnetic field processes with the FI-method were performed using Fourier-transformed frequency domain formulations.\(^{32-34}\) Instead of the time dependent grid voltages and facet fluxes now the complex-valued phasors of these vector variables are considered in the Maxwell–Grid-Equations. The time-harmonic curl–curl-equation of the electric grid tensions\(^{35}\) reads as

\[
[D + i\omega D_e - \omega^2 D_i]e = -i\omega j_e
\] (30)

This system of equations is always indefinite due to the negative contribution of \( \omega^2 D_e \) to the main diagonal. With the relation \( e = -i\omega a \) corresponding to equation (11), the equation (30) may also be expressed in terms of the complex phasor of the modified magnetic vector potential \( a \).

In case of the excitation frequency \( \omega \) coinciding with a modal frequency the equation itself will no longer be regular and thus no longer has a unique solution. In practice this equation (30) however is well suited for the calculation of time-harmonic electromagnetic fields over a wide range of possible excitation frequencies.\(^{16}\) Since also the displacement currents \(-i\omega D_e \) are taken into consideration in equation (30), it allows the simulation of high frequency problems which include wave propagation phenomena as well as the calculation of magneto-quasistatic problems, where the displacement currents will make no significant contribution to the system matrix if metallic conductors have to be considered. In this case one has to deal with a system which is mainly governed by the singular curl–curl-part which has only non-negative eigenvalues and where a purely imaginary valued contribution to the main diagonal is made in form of the usually singular diagonal matrix \( i\omega D_e \). Thus basically a singular, complex symmetric linear system has to be solved in the case of slowly varying electromagnetic fields.

A different modelling is necessary for high voltage fields, where the solenoidal parts of the electric field may be neglected in the simulation, but where dielectric fluxes and small ohmic losses have to be considered. These assumptions may have to be considered for high voltage problems at low frequencies modelled from isolators with small conductive contaminations,\(^{36-38}\) with high voltage transmission lines\(^{39}\) and in simulations of semiconductor elements.\(^{40}\) These so-called electro-quasistatic assumptions can be modelled using a discrete Helmholtz-decomposition for the electric grid voltages \( e = -\frac{1}{i\omega} a - \tilde{S}^t\Phi \), assuming the rotational parts represented by

\[
-\frac{d}{dt} a := 0
\] (31)

to vanish. With \( b = Ca \) the time variation of the magnetic fluxes \( \frac{d}{dt} b = 0 \) is assumed to be zero.

The governing equation of electro-quasistatics for the time-harmonic electric scalar potential follows with application of the discrete divergence \( \tilde{S} \) to (4) such that

\[
\tilde{S}(D_e + i\omega D_i)\tilde{S}^t\Phi = \tilde{S}j_e
\] (32)
holds. The system matrix in this complex-valued, discrete potential problem consists of the two parts

\[ A_i := \tilde{S}D_i\tilde{S}^T, \quad A_e := \tilde{S}D_e\tilde{S}^T \] (33)

where \(A_i\) is the real-valued symmetric, positive semi-definite matrix of stationary currents and \(A_e\) is the real-valued, symmetric, positive definite matrix of electrostatics of the FI-method.\(^{37}\)

### 4.1. Iterative solution methods for complex-symmetric problems

For both quasistatic frequency domain formulations for slowly varying electromagnetic fields, where either the displacement currents or the eddy currents will be omitted, the solution of a complex-valued, non-Hermitean linear system becomes necessary. For these problems general Krylov subspace methods are applicable.\(^{41}\) Such methods do not exploit the symmetry of the systems. In Reference 42 for the Bi-Conjugate-Gradient-method (BiCG)\(^{43}\) a whole class of so-called SCBICG-variants for symmetric, complex-valued problems was shown to exist. These methods will exploit the symmetry of the matrices and require only one matrix vector multiplication in each iteration step instead of the two matrix vector multiplications in the BiCG-method for general systems. The methods cover the well-known COCG-method\(^{44}\) and the new BiCGCR-method, which will coincide with the Conjugate-Residual-method (CR)\(^{45}\) for real-valued symmetric matrices, for which it will minimize the expression \(\|b - Ax_k\|_2\) in each iteration step. In the BiCG-methods the system matrix \(A\) is obliquely projected onto the constructed vector basis \(V_k := [v_1, \ldots, v_k]\) with \(V^TAV = T_{kk}\), where \(T_{kk}\) is a tridiagonal matrix. If \(T_{kk}\) becomes singular for some values of \(k\), the internal LU-decomposition of the matrix \(T_{kk}\) may break down and with it the whole iteration method.

Theoretically more robust methods are available with the Quasi-Minimal-Residual-methods for complex symmetric systems.\(^{46,47}\) These methods avoid such a breakdown situation, because they internally treat the tridiagonal matrices \(T_{kk}\) within a least-squares problem, for which a solution always exists. They may be made even more robust by introducing the so-called look-ahead-steps, which however is rarely really necessary in practice.

For the non-Hermitean, complex symmetric potential problems of electro-quasistatics of the FI-method these methods are compared in Reference 42. Since these methods are all based on the same BiCG-method, the convergence of the methods in the relative residual norm is similar with slight advantages for the BiCGCR-method due to its close relation to the CR-method which minimizes the residuals in the Euclidean norm.

The same Krylov subspace methods can be applied for the time-harmonic magneto-quasistatic eddy-current formulation (30). For such methods one might expect that the well-established General-Minimal Residual-method (GMRES)\(^{48}\) and its earlier (and more expensive) related variants GCR and Orthomin\(^{49}\) work well, since their iteration processes yield approximate solutions \(x_k\) which minimize the expression \(\|b - Ax_k\|_2\) in each iteration step. This corresponds to a least-squares solution of the problem. However, Krylov subspace methods featuring a minimal residual property for non-Hermitean matrices require the whole Krylov subspace basis to be stored during the iteration process. This causes almost linearly growing costs of storage with each iteration. Thus truncated variants such as GMRES(m) have to be considered in practical calculations, which are stopped after a basis of \(m\) Krylov subspace vectors and restarted with the old solution as initial guess. In connection with the SSOR-preconditioner these minimal-residual methods are not competitive when compared to the specialized BiCG-type methods. Future
developments for preconditioning methods however may render the GMRES-method feasible again.

5. Applications

5.1. The shading ring sensor

The results for the Shading rings sensor developed by the company Robert Bosch GmbH, Stuttgart, Germany, obtained by the FI-Methods for the frequency domain solution of equation (30) have been published earlier in References 16, 35 and 50. The sensor consists of an E-shaped iron core with an assumed constant relative permeability \( \mu_r = 1500 \) channeling a time-harmonic magnetic flux \( \mathbf{B} \). The flux is excited at 10 kHz by a copper coil placed at the base of the middle branch of the E-shaped iron yoke. It induces an eddy current in the second copper shading ring which induces a magnetic flux in the opposite direction of the flux from the excitation coil, thus forcing the flux to vanish behind the shading coil. The shading coil is constructed to allow a varying position along the iron yoke. The arrowplot in Figure 2 shows the real part of the magnetic flux in a quarter of a model discretized with 18,000 gridpoints. Calculations could be performed with the frequency formulation (30) and the time domain formulation, for which the magneto-quasistatic formulation (12) was time integrated with a ramped sinusoidal current excitation for a time interval \( 0 \leq t \leq 2.5 \times 10^{-6} \) s, corresponding to 1-5 periods at 10 kHz with 40 equidistant timesteps per period. This system was solved using the first-order BDF1-method and the second-order schemes BDF2, trapezoidal-rule and the stiffly accurate SDIRK2-scheme. Figure 3 shows the almost linear relation between the measured values\(^{35}\) and the good agreement of the calculated values for both the frequency and the magneto-quasistatic time domain solution for the real part of the inductivity depending on the distance of the two coils (Figure 4).

For the solution of the frequency domain system (30) a number of different Krylov subspace methods has been tested. Figure 5 shows the results for the convergence patterns of the SCBiCG-variants COCG and BiCGCR and two QMR-methods (2/3-term) in comparison to a GMRES(25)-method, which was restarted after 25 steps. For all these methods an implicit SSOR-preconditioning was applied. The BiCG-variants show about the same rates of overall convergence for this problem and even feature the typical superlinear convergence. This property is lost for the restarted GMRES-method, which is not competitive even for this small problem.

Figure 6 shows the industrial design of a shading ring sensor of the Robert Bosch GmbH\(^{54}\) based on the same principle as the E-shaped structure in Figure 2. The coil on the left is designed merely to control temperature effects due to ohmic losses. The whole structure was simulated using the symmetry in \( z \)-direction with \( 89 \times 64 \times 15 = 85440 \) grid nodes. The resulting dimension of the systems to be solved in time and frequency domain was 256 320 unknowns. The frequency domain solution was achieved using the SSOR-BiCGCR-(MRS)-method for equation (30) after 10,000 iterations and 11 h 40 min calculation time on a SUN Ultra 2 (167 MHz) at the end of which a relative residual of \( \| r_k \|_2 / \| r_0 \|_2 = 4.03 \times 10^{-6} \) was reached.

The relative accuracy with respect to the frequency domain solution of the magneto-quasistatic transient solution using a ramped sinusoidal excitation for different numbers of timesteps per period is shown in Figure 7. It has to be noted that a transient solution with 20 timesteps per period yields a relative error of less than 1 per cent in simulation times comparable to that of the reference frequency domain solution.
Figure 2. In the left picture the real part of the magnetic field \(\text{Re}(H)\) is shown in the model of a \(E\)-shaped shading ring sensor. On the right the calculated magnetic flux \(\text{Re}(B)\) in a quarter structure is shown for a distance between the current excitation coil and the shading ring of 16 mm. Both arrowplots feature a logarithmic scaling of the calculated field to show the effect of the shading coil on the right.

Figure 3. The real part of the inductivity \(L\) is plotted for different distances of the excitation and the shading coil in Figure 2.

5.2. The TEAM Problem 11

In the TEAM workshop problem 11 a hollow sphere with an outer radius of 0.055 m and an inner radius of 0.05 m, a conductivity of \(5 \times 10^8\) S/m and a relative permeability of 1.0 is abruptly exposed to a unidirectional, uniform magnetic field of 1 Tesla. For this problem a magneto-quasistatic solution is given, e.g. in References 52 and 53. A time discrete solution is achieved with the time integration of the differential-algebraic equation (12), using 40 equidistant timesteps for a time interval \(0 \leq t \leq 20\) ms. Figure 8 shows the discretized octant of...
Figure 4. The relative error of the frequency and time domain solutions is presented for the calculated real parts of the inductivity at the different distances between the two coils. The first-order BDF1 shows the largest errors from the measured values, whereas the second-order BDF2 method exhibits the lowest error of the magneto-quasistatic time domain methods.

Figure 5. Convergence history of the SSOR-preconditioned SCBiCG-methods BiCGCR and COCG, both equipped with minimal residual smoothing (MRS), the QMR-methods as 2-term- and 3-term-variants and the preconditioned GMRES(25)-method for the solution of the timeharmonic curl–curl equation (30) modelling the quarter structure of the shading ring sensor in Figure 2.

the problem geometry and a comparison of the analytical solution with the solutions obtained by the BDF1-, the BDF2- and the SDIRK2-methods, at a time $t = 20\text{ ms}$ after the solenoidal outer magnetic fields was switched on.
This problem nicely allows to show some difficulties occurring with the trapezoidal-rule time integration scheme when applied to problems where abrupt changes in the excitation term occur. Figure 9 compares the solutions of the BDF2-method and of the trapezoidal-rule at a time $t = 1\text{ ms}$ after the exterior magnetic field was switched on. The trapezoidal-rule does not damp out the spurious, oscillatory solution parts belonging to small time constants and thus the obtained field solution becomes erroneous. The oscillatory solution parts appear due to the abrupt change in the boundary conditions at $t = 0\text{ s}$.
Figure 8. For the hollow sphere of the TEAM 11 problem only an octant is discretized. The current strips on the other side are used to excite a solenoidal outer magnetic field $B_z$ of 1 Tesla starting at $t = 0\,\text{s}$. The current strips on the other side are used to excite a solenoidal outer magnetic field $B_z$ of 1 Tesla starting at $t = 0\,\text{s}$. The diagram on the right compares the analytic solution of the TEAM 11 problem with the solution of the stiffly accurate one-step time integration methods $BDF1$ and $SDIRK2$ and the $BDF2$-multistep-method at $t = 20\,\text{ms}$. The difference of all the implicit time domain solutions to the analytical solution is due to an inexact discretization of the sphere curvature in the Cartesian grid and a rather coarse mesh on the outside of the coil.

Figure 9. The arrowplot on the left shows the magnetic flux $B$ calculated with the $BDF2$-method at 1 ms at the position $y = z = 0\,\text{m}$ along the $x$-axis. The right plot shows a solution obtained with the trapezoidal-rule which shows no numerical damping of high-frequency solution parts occurring due to the abrupt change in the external magnetic field at $t = 0\,\text{s}$.

6. Conclusions

In this paper we presented formulations based on the FI-method and the corresponding numerical methods for the simulation of a slowly varying electromagnetic field in time and frequency domain. For such time domain problems a reduced-$c_0$-approach allows to use the
otherwise unfeasible standard FDTD method for problems with linear material properties. A transient magneto-quasistatic formulation based on a modified vectorpotential defined in the whole computational domain is presented featuring the required gauging techniques. The resulting differential-algebraic problems of index 1 have to be solved using implicit time-integration schemes. Apart from the standard $\theta$-time-integration schemes, suitable methods of higher order are available with stably accurate $SDIRK$-s-stage one-step-methods and the implicit multistep $BDF(k)$-formulas of Gear ($k < 7$). For these implicit time discretization schemes, preconditioned conjugate gradient methods are used for the algebraic systems of equations. In time domain schemes usually symmetric, positive semi-definite systems have to be solved at least once in each timestep. In frequency domain formulations a non-Hermitean system has to be solved once, for which efficient Bi-Conjugate-Gradient-related methods are best suited. These schemes yield meaningful solutions for magneto-(quasi)-static formulations for which no explicit gauging is formulated. This is due to the consistency of the discretization method and the weakly imposed gauging properties of Conjugate-Gradient-type methods. An example of an E-shaped shading ring sensor shows good agreement to measurements for both numerical approaches. Both methods allow also to analyze a larger, more complicated industrial design. Here the advantage of the real-valued formulations of the transient approach for very large problems becomes apparent. The TEAM 11 workshop problem was used to show the difficulties that may arise from using the second-order trapezoidal-rule without numerical damping for the time magneto-quasistatic differential-algebraic systems of equations with index 1.

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References

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