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TOKAMAK PLASMA SHAPE IDENTIFICATION
ON THE BASIS OF BOUNDARY INTEGRAL EQUATIONS

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ABSTRACT. The mathematical concepts of the necessary conditions for tokamak plasma shape identification are discussed. A method that uses only the derived necessary condition is proposed. This method is based on the boundary integral equations governing the vacuum region around the plasma, using only the measured values of either the magnetic fluxes or the magnetic flux intensities. The application to JT-60U and ITER plasmas shows that this method can be used to identify various plasmas with low to high ellipticities, with the necessary precision, when an adequate number of magnetic sensors is provided. The proposed method is also applicable to real-time control and visualization.

1. INTRODUCTION

To perform experiments and data evaluation in tokamak devices, plasma shape identification is needed for plasma equilibrium control and analysis. Various methods for this identification have been proposed [1]. The method called filament current approximation [2] seems to be the one most frequently applied to tokamaks. This method uses several filament current coils to obtain the vacuum magnetic field produced by the plasma current. Although it is based on a simple approximation, this method reproduces the plasma shape comparatively well using only magnetic measurements. In contrast, methods based on an exact analytical solution of partial differential equations (PDEs), such as the Legendre–Fourier expansion (LFE) [3] and the 'multipole moment expansion' [4, 5], give poor identification for plasmas with certain shapes or current profiles [3]. This is due to the small number of sensors employed and also to the numerical limitations, which only permit a calculation of the first few series in solution formulas composed of an infinite series of eigenfunctions. Another method, based on an exact analytical solution of PDEs, is the 'boundary integral equation' method (see Appendix 1). This application was reported by Hakkarainen and Freidberg [6], who used two kinds of sensors, corresponding to the Dirichlet and Neumann conditions on the boundary, with several poloidally nested hypothetical contours. For full equilibrium analysis [7], with formulas for the plasma current and pressure profiles, the plasma shape must be known. The premise that the predetermined formulas for this analysis match the various plasma states is still questionable. Furthermore, the simple application of the LFE does not accurately reproduce highly elongated plasmas ($\kappa = 2.2$) such as that of the International Thermonuclear Experimental Reactor (ITER) [8, 9]. The reason for this is not clear; it seems that the process of plasma shape identification is not fully understood.

In the present paper, the necessary condition for plasma shape identification is discussed on the basis of the exact analytical solution of PDEs (a method using boundary integral equations), from the point of view of conceptual mathematics and numerical computation. It is confirmed that the method utilized for the derivation of the condition gives the shape reproduction with only the necessary condition as applied to JT-60 Upgrade (JT-60U) and ITER.

Conceptual and theoretical considerations of shape identification are discussed in Section 2. Some techniques of the numerical computation used are described in Section 3. The application to plasmas in JT-60U and ITER is discussed in Section 4, and conclusions are given in Section 5.

2. TOKAMAK PLASMA SHAPE IDENTIFICATION CONCEPT

It is well known that a tokamak plasma is so light that it is considered to preserve its equilibrium state. Therefore, the problem is to identify the static magnetic field that results from currents flowing in both the plasma and the poloidal field (PF) coils. Magnetic measurement data are used for this identification because
the magnetic sensors in tokamaks are considered to be quite reliable at present and probably also in the future. These sensors measure the magnetic or flux fields in the vacuum around the plasma. Since the plasma surface is a border surface of the vacuum region, complete identification of the magnetic fields in this vacuum region would provide an identification of the plasma shape. Open questions are whether this is possible in an actual system and what kinds of magnetic sensors are necessary. Answers to these questions are discussed below.

2.1. Topological concept of the problem and formulation of the equations

The current density \( \vec{j} \) and the magnetic flux intensity \( \vec{B} \) are connected by the following static Maxwell’s equations:

\[
\text{rot } \vec{B} = \mu_0 \vec{j} \quad (2.1)
\]

\[
\text{div } \vec{B} = 0 \quad (2.2)
\]

By introducing the vector potential \( \vec{A} \) (\( \vec{B} = \text{rot } \vec{A} \)) from Eq. (2.2), Eq. (2.1) becomes

\[
\text{rot } \text{rot } \vec{A} = \mu_0 \vec{j} \quad (2.3)
\]

It is assumed that all quantities are axisymmetric. The flux function is defined as \( \phi = r A_\phi \), where \( A_\phi \) is the toroidal component of the vector \( \vec{A} \), and \( r \) is the distance from the axis in cylindrical co-ordinates. Then, Eq. (2.3) can be converted to the following scalar equation:

\[
\text{div} \left[ \frac{\text{grad } \phi}{r^2} \right] = -\frac{\mu_0 j_\phi}{r} \quad (2.4)
\]

where \( j_\phi \) is the toroidal component of the vector \( \vec{j} \). The variable \( j_\phi \) can be expressed by using the pressure and current profile functions, \( p(\phi) \) and \( I(\phi) \), in terms of the flux function \( \phi \):

\[
j_\phi = \frac{\mu_0 r}{2} \frac{d^2}{d\phi^2} + \frac{r dp}{d\phi} \quad (2.5)
\]

The substitution of the right hand side of Eq. (2.4) by Eq. (2.5) yields the Grad–Shafranov equation. For the vacuum region, \( j_\phi \) is set to zero in Eq. (2.4), and the following equation is obtained:

\[
\text{div} \left[ \frac{\text{grad } \phi}{r^2} \right] = 0 \quad (2.6)
\]

This equation is classified as an elliptic non-linear homogeneous second-order PDE. In general, a boundary value problem for an elliptic second-order PDE is known to give a unique solution with a Dirichlet condition or a Neumann condition on a closed boundary [10]. However, since the plasma current profile is unknown, the analytical region concerned is a doughnut shaped area surrounding the plasma. (This topological concept is shown in Fig. 1.) It seems to be unclear whether the uniqueness of solution of a general elliptic second-order PDE still holds in a region that is not simply connected. However, the analytical solution of Eq. (2.6), based on the method of separation of variables, gives two kinds of eigenfunction, one of which has a singular point [3]. This suggests the existence of a unique solution for Eq. (2.6) in a region that is not simply connected, with a Dirichlet condition or a Neumann condition on a closed boundary. In fact, the shape identification method using these eigenfunctions gives a good result for low-\( \kappa \) plasmas [3].

Now, Eq. (2.4) is solved by converting it to the form of an integral equation using a Green’s function because this approach is most advantageous for a numerical solutions of the concerned PDE. (The reason for this selection is explained in Appendix 1.) In the identity of the scalar functions \( f \) and \( g \):

\[
\text{div} \left[ \frac{f \cdot (\text{grad } g)}{r^2} \right] - \text{div} \left[ \frac{g \cdot (\text{grad } f)}{r^2} \right] = f \cdot \text{div} \left[ \frac{\text{grad } g}{r^2} \right] - g \cdot \text{div} \left[ \frac{\text{grad } f}{r^2} \right] \quad (2.7)
\]

\( \phi(\vec{Y}) \) and \( G(\vec{X}, \vec{Y}) \) are substituted for \( f \) and \( g \), respectively. The function \( G \) is the Green’s function between the two points, \( \vec{X} \) and \( \vec{Y} \), in an axisymmetric geometry.
The points $\mathbf{x}$ and $\mathbf{y}$ are defined as $(r_x,z_x)$ and $(r_y,z_y)$, respectively, in cylindrical co-ordinates. The function $G$ is then expressed as

$$G(x,y) = G(r_x,z_x,r_y,z_y) = \frac{4}{k} \frac{\sqrt{r_xr_y}}{r_xr_y} \left[ \left( 1 - \frac{k^2}{2} \right) K(k) - E(k) \right]$$

(2.8)

where $K$ and $E$ are the complete elliptic integrals of the first and second kinds, and $k^2 = 4r_xr_y/((r_x + r_y)^2 + (z_x - z_y)^2)$. Furthermore, since this Green's function is identical with the zeroth order eigenfunction of Eq. (2.6), the following equation holds:

$$\text{div} \left[ \frac{\text{grad} G(\mathbf{x},\mathbf{y})}{r_y^2} \right] = \gamma \cdot \delta(\mathbf{x},\mathbf{y})$$

(2.9)

$$\gamma \equiv \lim_{x \to 0} \int_{\Omega} \text{div} \left[ \frac{\text{grad} G(\mathbf{x},\mathbf{y})}{r_y^2} \right] \cdot dV(\mathbf{y}) = -8\pi^2$$

(2.10)

where $\delta(\mathbf{x},\mathbf{y})$ is the delta function, and $dV(\mathbf{y})$ is the infinitesimal volume element at point $\mathbf{y}$. After integrating the identity Eq. (2.7) in the volume $\Omega$ with respect to $\mathbf{y}$, Eqs (2.4), (2.9), (2.11) and the Gauss integral formula are taken into account, with $\Omega \supset \mathbf{x}$. Then the solution of Eq. (2.4) is obtained:

$$\sigma \phi(\mathbf{x}) = \int_{\partial \Omega} [G(\mathbf{x},\mathbf{y}) \cdot \text{grad} \phi(\mathbf{y}) - \phi(\mathbf{y}) \cdot \text{grad} G(\mathbf{x},\mathbf{y})] \cdot \frac{d\mathbf{s}(\mathbf{y})}{r_y^2} + \int_{\Omega} \mu_0 j(\mathbf{y}) \cdot G(\mathbf{x},\mathbf{y}) \cdot \frac{dV(\mathbf{y})}{r_y}$$

(2.12)

where $\partial \Omega$ is the closed surface of the volume $\Omega$, $d\mathbf{s}(\mathbf{y})$ is the infinitesimal surface element vector at point $\mathbf{y}$, $\partial \Omega$ is the vector normal to the surface $\partial \Omega$ in the direction away from $\Omega$, $j(\mathbf{y})$ is the current density distribution involved in the region $\Omega$, and $\sigma$ is a constant defined by $\sigma = -\gamma \theta$, where $\theta = \{1(\Omega \supset \mathbf{x}), 0(\Omega \cap \mathbf{x}) \}$ [11], 0 (area excluding $(\Omega + \partial \Omega) \supset \mathbf{x})$. Equation (2.12) signifies that if the current distribution in the volume $\Omega$ is known, the condition for tokamak plasma shape identification using this solution is discussed in the following section.

### 2.2. A necessary condition for plasma shape identification

Equation (2.12) requires a knowledge of $j(\mathbf{y})$ in the volume $\Omega$. Since the plasma current distribution was not known at the time of shape identification, the plasma current cannot be taken as $j(\mathbf{y})$ for Eq. (2.12). The distributions of the PF coil current and of other currents, except the plasma current, are assumed to be known. This does not degrade the conceptual discussion of shape identification. The reason for this assumption is that the field produced by unobservable currents such as eddy currents is negligibly small compared with the field produced by the plasma current. An exception is the case of short periods of current buildup and disruption [12].

Three closed curved surfaces, nested within one another, are now defined in the axisymmetric analytical region. On the poloidal cross-section, there are nested closed curved surfaces, as illustrated in Fig. 1. The first surface is the boundary surface, $\partial \Omega_b$, which contains the whole region of concern. The second surface is the sensor surface, $\partial \Omega_s$, along which the sensors are located and which is enclosed by $\partial \Omega_b$. The third surface is the hypothetical plasma surface, $\partial \Omega_p$, which encloses the plasma column and is enclosed by $\partial \Omega_b$. The area bounded by the surface $\partial \Omega_b$ and $\partial \Omega_s$ is denoted by $\Omega_{b,s}$. Equation (2.12) is applied to two regions, $\Omega_{b,s}$ and $\Omega_{p,s}$, which do not involve the plasma. Subsequently, $\partial \Omega_b$ is moved infinitely far away from the area concerned. With respect to $\mathbf{y}$ on the central axis, $G(\mathbf{x},\mathbf{y}) = 0$ and $\text{grad} G(\mathbf{x},\mathbf{y}) = 0$. By taking into account the following limitation:

$$\lim_{\partial \Omega_b \to \infty} \int_{\partial \Omega_b} [G \text{ grad } \phi - \phi \text{ grad } G] \cdot \frac{d\mathbf{s}(\mathbf{y})}{r_y^2} = 0$$

(2.13)

the surface integral on $\partial \Omega_b$ in Eq. (2.12) vanishes. Consequently, the solutions for Eq. (2.4) in the regions $\Omega_{b,s}$ and $\Omega_{p,s}$ are as follows:

For $\mathbf{x} \subseteq \Omega_{b,s}$,

$$\sigma \phi(\mathbf{x}) = \int_{\partial \Omega_s} [G \text{ grad } \phi - \phi \text{ grad } G] \cdot \frac{d\mathbf{s}(\mathbf{y})}{r_y^2} + \int_{\partial \Omega_{b,s}} \mu_0 j \cdot G \cdot \frac{dV(\mathbf{y})}{r_y}$$

(2.14)

For $\mathbf{x} \subseteq \Omega_{p,s}$,

$$\sigma \phi(\mathbf{x}) = \int_{\partial \Omega_p} [G \text{ grad } \phi - \phi \text{ grad } G] \cdot \frac{d\mathbf{s}(\mathbf{y})}{r_y^2}$$

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By using Eq. (2.14), again for \( \mathbf{v} \neq \mathbf{0} \),
\[
\sigma \mathbf{v} = \int_{\partial \Omega_p} \left[ G \, \text{grad} \, \phi - \phi \, \text{grad} \, G \right] \cdot \frac{dS(y)}{r_y^2} 
+ \int_{\partial \Omega_p} \mu_0 j_v \, G \, \frac{dV(y)}{r_y} 
\tag{2.15}
\]

By using Eq. (2.14), again for \( \mathbf{v} \neq \mathbf{0} \),
\[
\sigma \mathbf{v} = \int_{\partial \Omega_p} \left[ G \, \text{grad} \, \phi - \phi \, \text{grad} \, G \right] \cdot \frac{dS(y)}{r_y^2} 
+ \int_{\partial \Omega_p} \mu_0 j_v \, G \, \frac{dV(y)}{r_y} 
\tag{2.16}
\]

In these equations, \( j_v \) and \( j_v \) are the known current densities in the regions \( \Omega_{n-p} \) and \( \Omega_{s-p} \), respectively. For \( dS(y) = \mathbf{n}(y) \cdot dS(y) \), \( \mathbf{n} \) is the vector normal to the surface \( \partial \Omega_n \) or the surface \( \partial \Omega_p \) in the direction towards the plasma.

The integral along the curved surface \( \partial \Omega_n \) in Eq. (2.14) requires that the values of \( \phi(y) \) and \( \text{grad} \, \phi \) be measured. The quantity \( \phi(y) \) is directly measured by the time integral of the voltage signal from a one-turn flux loop. The quantity \( \mathbf{B} \) denotes a tangential component of a flux intensity vector to the curved surface \( \partial \Omega_n \), and \( \mathbf{B} \) can be expressed in terms of \( \phi \) as \( \mathbf{B} = \mathbf{n} \times \nabla \phi \). The signal from a magnetic probe located along \( \partial \Omega_n \) gives \( \nabla \phi \). Therefore, \( \phi \) and \( \mathbf{B} \) can be measured by magnetic sensors.

An infinite series formula is introduced for further discussion. \( K(\alpha, \beta) \) is a bounded and smooth function, except in the case where \( \alpha = \beta \), and \( q(\beta) \) is a smooth function in the integral domain \( \Omega \). If \( \alpha \) and \( \beta \) are involved in \( \Omega \), and if the integral of \( K(\alpha, \beta) \) with respect to \( \beta \) is bounded, i.e. \( K(\alpha, \beta) \, d\beta \ll \infty \), then
\[
\int_a^b K(\alpha, \beta) \, q(\beta) \, d\beta = \sum_{i=1}^{m} w(h_i, \alpha, \beta_i) \, q(\beta_i) \tag{2.17}
\]
(see Appendix 2) where \( h_i \) is the \( i \)-th infinitesimal length for the definite integral in the interval \( [a, b] \). Using a sufficiently large number \( N \) for the division of the integral interval, Eq. (2.17) can be converted to
\[
\int_a^b K(\alpha, \beta) \, q(\beta) \, d\beta = \sum_{i=1}^{N} w(h_i, \alpha, \beta_i) \, q(\beta_i) + \delta(\alpha, N) \tag{2.18}
\]
where \( \delta(\alpha, N) \) is a sufficiently small function compared with the first term on the right hand side of Eq. (2.18).

This function is presumed to decrease monotonically as \( N \) increases, i.e. \( \delta(\alpha, N \to \infty \to 0) \).

Now, we take the numbers for the division of the curved surfaces \( \partial \Omega_n \) and \( \partial \Omega_p \) as \( N \) and \( M \), respectively, and \( N \geq M \). Then, Eqs (2.14)-(2.16) are expressed as infinite series formulas by utilizing Eq. (2.18), where the residual function \( \delta \) denotes the total amount of residues produced by all integrals in Eqs (2.14), (2.15) or (2.16). In Eq. (2.14), by bringing \( \mathbf{v} \) infinitely close to the boundary \( \partial \Omega_n \) from the inside, we obtain
\[
\phi(\mathbf{v}) = \sum_{i=1}^{N} W_i(\mathbf{v}, \mathbf{y}_i) \cdot \phi(\mathbf{y}_i) 
+ \sum_{i=1}^{N} W_i(\mathbf{v}, \mathbf{y}_i) \cdot \mathbf{B}_i(\mathbf{y}_i) + W_i(\mathbf{v}) + \delta(\mathbf{v}, N) \tag{2.19}
\]
with \( \mathbf{v} \neq \partial \Omega_n \) and \( \mathbf{y}_i \neq \partial \Omega_n \).

Similarly, in Eq. (2.16), \( \mathbf{v} \) is brought infinitely close to \( \partial \Omega_p \) and \( \mathbf{y}_i \) from the inside, and then we obtain
\[
\phi(\mathbf{v}) = \sum_{i=1}^{M} W_i(\mathbf{v}, \mathbf{z}_i) \cdot \phi(\mathbf{z}_i) 
+ \sum_{i=1}^{M} W_i(\mathbf{v}, \mathbf{z}_i) \cdot \mathbf{B}_i(\mathbf{z}_i) + W_i(\mathbf{z}) + \delta(\mathbf{z}, M) \tag{2.20}
\]
with \( \mathbf{v} \neq \partial \Omega_p \) and \( \mathbf{z}_i \neq \partial \Omega_p \).

Equation (2.19) (or Eq. (2.14)) expresses the relation between \( \phi \) on \( \partial \Omega_n \) and \( \mathbf{B}_i \) on \( \partial \Omega_n \). Equation (2.20) expresses the relation between \( \phi \) on \( \partial \Omega_p \) and \( \phi \) and \( \mathbf{B}_i \) on \( \partial \Omega_p \). Equation (2.21) expresses the relation between \( \phi \) on \( \partial \Omega_p \) and \( \mathbf{B}_i \) on \( \partial \Omega_n \). Since Eqs (2.19)-(2.21) are linear, they can be composed as vector equations with matrix coefficients. According to the existence theorems for the solution of the Fredholm integral equation of the first kind, no solution exists, except for the special case in analytical discussions. However, no difficulty is encountered in the discretized Fredholm equation.

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1 Classification of the integral equation types \( X \neq \partial \Omega_n \) in Eq. (2.14) and \( X \neq \partial \Omega_p \) in Eq. (2.16) yields the boundary integral equations. These belong to the Fredholm integral equation of the second kind for the flux function \( \phi \) to that of the first kind for the flux intensity \( \mathbf{B} \). The relation \( X \neq \partial \Omega_n \) in Eq. (2.16) is the Fredholm integral equation of this first kind for both \( \phi \) and \( \mathbf{B} \) on \( \partial \Omega_n \). The Fredholm equation of the second kind has a unique solution, while that of the first kind does not have a solution except in a special case [13].
With \( \mathbf{x} = \mathbf{y}_i \) (i = 1, 2, ..., N) in Eqs (2.19) and (2.20), two sets of simultaneous N equations are obtained. Similarly, with \( \mathbf{x} = \mathbf{z}_i \) (i = 1, 2, ..., M) in Eq. (2.21), a set of simultaneous M equations is obtained. The three sets of the vector equations are then expressed as:

\[
\mathbf{\varphi}^a = \mathbf{\bar{A}}_1 \mathbf{\varphi}^a + \mathbf{\bar{B}}_1 \mathbf{\varphi}^a + \mathbf{c}_1 + \mathbf{d}_1
\]

(2.22)

\[
\mathbf{\varphi}^b = \mathbf{\bar{A}}_2 \mathbf{\varphi}^b + \mathbf{\bar{B}}_2 \mathbf{\varphi}^b + \mathbf{c}_2 + \mathbf{d}_2
\]

(2.23)

\[
\mathbf{\varphi}^c = \mathbf{\bar{A}}_3 \mathbf{\varphi}^c + \mathbf{\bar{B}}_3 \mathbf{\varphi}^c + \mathbf{c}_3 + \mathbf{d}_3
\]

(2.24)

where the vectors and matrix coefficients are defined as follows:

\[
\mathbf{\varphi}^a = \begin{bmatrix} \phi_1(y_1), \phi_2(y_2), \phi_3(y_3), \ldots, \phi(N) \end{bmatrix} \in \mathbb{R}^{N \times 1}
\]

\[
\mathbf{\bar{A}}_1 = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1N} \end{bmatrix} \in \mathbb{R}^{N \times N}
\]

\[
\mathbf{\bar{B}}_1 = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1N} \end{bmatrix} \in \mathbb{R}^{N \times M}
\]

\[
\mathbf{c}_1 = \begin{bmatrix} c_1 & c_2 & \ldots & c_N \end{bmatrix} \in \mathbb{R}^{N \times 1}
\]

\[
\mathbf{d}_1 = \begin{bmatrix} d_1 & d_2 & \ldots & d_N \end{bmatrix} \in \mathbb{R}^{N \times 1}
\]

\[
\mathbf{\varphi}^b = \begin{bmatrix} \phi_1(y_1), \phi_2(y_2), \phi_3(y_3), \ldots, \phi(M) \end{bmatrix} \in \mathbb{R}^{M \times 1}
\]

\[
\mathbf{\bar{A}}_2 = \begin{bmatrix} a_{21} & a_{22} & \ldots & a_{2M} \end{bmatrix} \in \mathbb{R}^{M \times M}
\]

\[
\mathbf{\bar{B}}_2 = \begin{bmatrix} b_{21} & b_{22} & \ldots & b_{2M} \end{bmatrix} \in \mathbb{R}^{M \times M}
\]

\[
\mathbf{c}_2 = \begin{bmatrix} c_1 & c_2 & \ldots & c_M \end{bmatrix} \in \mathbb{R}^{M \times 1}
\]

\[
\mathbf{d}_2 = \begin{bmatrix} d_1 & d_2 & \ldots & d_M \end{bmatrix} \in \mathbb{R}^{M \times 1}
\]

\[
\mathbf{\varphi}^c = \begin{bmatrix} \phi_1(z_1), \phi_2(z_2), \phi_3(z_3), \ldots, \phi(M) \end{bmatrix} \in \mathbb{R}^{M \times 1}
\]

\[
\mathbf{\bar{A}}_3 = \begin{bmatrix} a_{31} & a_{32} & \ldots & a_{3M} \end{bmatrix} \in \mathbb{R}^{M \times M}
\]

\[
\mathbf{\bar{B}}_3 = \begin{bmatrix} b_{31} & b_{32} & \ldots & b_{3M} \end{bmatrix} \in \mathbb{R}^{M \times M}
\]

\[
\mathbf{c}_3 = \begin{bmatrix} c_1 & c_2 & \ldots & c_M \end{bmatrix} \in \mathbb{R}^{M \times 1}
\]

\[
\mathbf{d}_3 = \begin{bmatrix} d_1 & d_2 & \ldots & d_M \end{bmatrix} \in \mathbb{R}^{M \times 1}
\]

By elimination of \( \mathbf{\bar{B}}^p \) from Eqs (2.23) and (2.24), using the method of least squares, the following relations are obtained:

\[
\mathbf{\varphi}^b = (\mathbf{\bar{F}}^p)^{-1} \begin{bmatrix} \mathbf{\bar{F}}^p \mathbf{\varphi}^a + (\mathbf{\bar{B}}_2 \mathbf{\bar{B}}_1^{-1} \mathbf{\varphi}^a + \mathbf{c}_2) + \mathbf{d}_2 \end{bmatrix}
\]

(2.25)

\[
\mathbf{\bar{F}} = \mathbf{\bar{A}}_1 + \mathbf{\bar{B}}_2 \mathbf{\bar{B}}_1^{-1} (\mathbf{\bar{F}} - \mathbf{\bar{A}}_3)
\]

where \( \mathbf{\bar{F}} \) is a unit matrix in \( \mathbb{R}^{N \times N} \)

\[
\mathbf{\bar{B}}^c = \mathbf{\bar{B}}_3^{-1} (\mathbf{\bar{F}} - \mathbf{\bar{A}}_3) \mathbf{\bar{B}}^p - \mathbf{\bar{B}}_1^{-1} \mathbf{\varphi}^a - \mathbf{\bar{B}}_1^{-1} \mathbf{c}_2 - \mathbf{d}_2
\]

(2.26)

Eq. (2.22) can be rearranged as

\[
\mathbf{\varphi}^a = (\mathbf{\bar{F}} - \mathbf{\bar{A}}_3)^{-1} (\mathbf{\bar{B}}^c + \mathbf{c}_1 + \mathbf{d}_1)
\]

(2.27)

where \( \mathbf{\bar{F}} \) is a unit matrix in \( \mathbb{R}^{N \times N} \)

The matrix inversions in Eqs (2.25), (2.26) and (2.27) can be calculated because the independence of the rows are preserved even for \( N \) and \( M \rightarrow \infty \) (preserving \( N \geq M \)). Since the vectors of the residuals \( \mathbf{d}_1, \mathbf{d}_2 \) and \( \mathbf{d}_3 \) monotonically converge to 0, \( \mathbf{\bar{B}}^p \) and \( \mathbf{\bar{B}}^c \) converge to the real values. By substitution of \( \mathbf{\bar{B}}^p \) and \( \mathbf{\bar{B}}^c \) into Eq. (2.16), a flux function value can be calculated at any point \( \mathbf{x} \) outside \( \partial \Omega_0 \). Equation (2.27) implies that \( \mathbf{\bar{B}}^p \) on \( \partial \Omega_0 \) gives \( \mathbf{\bar{B}}^c \) on \( \partial \Omega_0 \). Equations (2.25) and (2.26) imply that \( \mathbf{\bar{B}}^c \) on \( \partial \Omega_0 \) gives \( \mathbf{\bar{B}}^p \) on \( \partial \Omega_0 \). Therefore, the intermediate conclusion I results, which specifies that a necessary condition for identification of the flux function in a vacuum region outside the plasma (see Fig. 1) is that either the flux intensities tangential to the curved surface \( \partial \Omega_0 \) or the flux function values at the points on \( \partial \Omega_0 \) \( \mathbf{\bar{B}}^p \) must be continuously given. This conclusion confirms the result obtained in the Appendix of Ref. [2].

Conclusion I does not directly specify a condition for identifying the plasma shapes. However, it implies that the flux function values at all points in the vacuum region facing the plasma can be identified by magnetic measurements in a vacuum around the plasma. To advance the discussion, the properties of the outermost plasma surface have to be confirmed. It is well known that either a limiter on the first wall of the vacuum vessel or an X-point, which is usually produced by the divertor coil, determines the outermost flux surface having a certain value of the flux function. If the flux function distribution in the poloidal cross-section is compared to the 'geographical altitude of a mountain', then the contour having the highest (or lowest) altitude in a vacuum around a plasma identifies the outermost flux surface.

The hypothetical plasma surface \( \partial \Omega_0 \) has been located so that it encloses the plasma. Now, the solution of the integral equations by locating \( \partial \Omega_0 \) inside the plasma is considered. This corresponds to the identification of a different plasma that produces the same flux field outside a real plasma. Therefore, the identified flux field in the area between \( \partial \Omega_0 \) and a real plasma surface is no longer a reality. However, it can be proven that the identified flux field agrees with a real field outside a real plasma surface (see Appendix 3). As a result, a contour having the properties of the plasma surface in the region outside \( \partial \Omega_0 \) uniquely exists, and this contour is denoted by \( \partial \Omega_0 \). The quantities \( \mathbf{\bar{B}}^p \) and \( \mathbf{\bar{B}}^c \) denote

---

2 If there is more than one contour having plasma surface properties, one of them must be a real plasma surface. The remaining contours must be inside a real plasma because the exact flux field outside a real plasma has been solved. However, these remaining contours contradict the properties of the outermost plasma surface.
the flux function on $\partial \Omega_p$ and the flux intensity tangential to the curved surface $\partial \Omega_p$, respectively. It is clear that $\mathbf{B}_p$ and $\mathbf{E}_p$ are the result of the boundary integral equations where $\partial \Omega_p$ is an inner boundary surface — the hypothetical plasma surface. Thus, we obtain the intermediate conclusion II, which specifies that a closed surface having the properties of the plasma surface uniquely exists in the region outside the hypothetical plasma surface. Thus, we obtain the flux function on $\partial \Omega_p$ and the flux intensity tangential to the curved surface $\partial \Omega_p$, respectively. It is clear that $\partial \Omega_p$ be located within a plasma.

Considering the intermediate conclusions I and II, the following conclusion is finally reached: A necessary condition to identify the shape of the plasma surface is that either the flux intensities tangential to the curved surface $\partial \Omega_p$ or the flux function values at the points on $\partial \Omega_p$ (or lowest) flux value in a vacuum around a plasma shows the outermost flux surface of a plasma. Thus:

$$\phi(\mathbf{x}) = \mathbf{c}_f(\mathbf{x}, \partial \Omega_p) \cdot \mathbf{E}_p + \mathbf{c}_q(\mathbf{x}, \partial \Omega_p) \cdot \mathbf{B}_p$$

(3.5)

where $\mathbf{x}$ is an arbitrary observation point in the region $\Omega_{B-P}$, $\mathbf{c}_f(\mathbf{x}, \partial \Omega_p)$, and $\mathbf{c}_q(\mathbf{x}, \partial \Omega_p)$ are the vectors resulting from the discretization of the first term on the right hand side of Eq. (2.16), $\mathbf{I}_c$ is the current vector of coils or conductors in the region $\Omega_{B-P}$, and $\mathbf{c}_f(\mathbf{x})$ is the coefficient vector that connects $\mathbf{I}_c$ with the flux value at point $\mathbf{x}$.

Equation (3.5) shows that when $\partial \Omega_p$ is determined, the coefficient vectors depend only on $\mathbf{x}$. By preparing these vectors in tabular form, the flux function value $\phi(\mathbf{x})$ can be calculated using only the measurements of $\mathbf{I}_c$ and either $\mathbf{B}_p$ or $\mathbf{E}_p$. This implies that this method is applicable to real-time control and visualization.

To execute these three calculations, the integral of the interval including a singular point should be performed. The integrands of $G(\mathbf{x}, \mathbf{y})$ and grad $G(\mathbf{x}, \mathbf{y})$ have a singular point of $\mathbf{y} = \mathbf{x}$, but they are integrable in the sense of the Riemannian integral concept. Two methods can be considered. (a) An interval having a singular point is discretized into parts of the same length, taking care that none of the discretized points ever be a singular point. Using these points, simple numerical integration along this interval is executed. The Simpson integral with 19 discretized parts gives adequate accuracy (this will be shown in the following section). (b) The interval of integration is limited to the neighbourhood of the singular point, and the calculation of the analytical approximate integral is performed, as discussed in Appendix 2.
4. APPLICATION TO JT-60U AND ITER

The method based on the boundary integral equations (the BIE method) is applied to JT-60U and ITER, and its performance is evaluated in this section. The method of evaluation is as follows: (a) A test plasma is produced by a reliable equilibrium code, and the magnetic flux intensities at the sensor locations are calculated beforehand. (b) The plasma shape is reproduced by the BIE method using the magnetic flux intensities. (c) The identified shape and the value of the flux contour produced by the equilibrium code are plotted on the same axis for comparison.

The application of the BIE method to JT-60U high $\beta_p$ divertor plasmas with different current profiles is shown in Fig. 2. The closed curved surface for the sensors, $\partial \Omega_s$, is the vessel wall. Fifty B, sensors (measuring a component of the magnetic field tangential to $\partial \Omega_s$) were located along $\partial \Omega_s$ at the same interval with respect to the angle $\theta$ in polar co-ordinates. The same ellipse:

$$[R_m(\theta), Z_m(\theta) = [3.33 + 0.5 \cos \theta \text{ m}, 1.0 \sin \theta \text{ m}]$$

in cylindrical co-ordinates, was used as the hypothetical plasma surface $\partial \Omega_p$ for all identifications shown in Fig. 2. The number of discretized points that are independent of each other on $\partial \Omega_p$ was 25. These were located at the same interval with respect to the parameter $\theta$. Figure 2 illustrates that differences in plasma current profiles have little influence on the accuracy of the identification.

The application of the BIE method to ITER is shown in Fig. 3. The identification of the standard high $\beta_p$ double-null divertor plasma is presented in Fig. 3(a). Figures 3(b–e) show the influence of the changes with regard to (a). Figure 3(b) shows the influence of the $\beta_p$ change, Fig. 3(c) the influence of the hypothetical plasma surface $\partial \Omega_p$, Fig. 3(d) the influence of the sensor-plasma distance without noise and with ±2% noise, and Fig. 3(e) the influence of ±1% and ±2% noise. Forty-eight B, sensors were located symmetrically to the 'midplane' (the mirror surface plane for symmetry, i.e. random noise, which obeys the normal distribution whose standard deviation is x% of the real value, is applied to all the magnetic sensor (B,) signals, and the random noise with 0.5% standard deviation is applied to coil current signals.)
FIG. 3. Application of the BIE method to ITER plasmas.
TOKAMAK PLASMA SHAPE IDENTIFICATION

which is often the equatorial plane of the torus) along \( \partial \Omega_p \) at the same interval with respect to the parameter \( \theta \). The curved surface \( \partial \Omega_b \) is defined on the poloidal cross-section in cylindrical co-ordinates as:

\[
[R_b(\theta), Z_b(\theta)] = [6.0 + a \cos (\theta + \{\sin^{-1} (0.4) \sin (\theta)\}) m, a \kappa \sin \theta m]
\]

The parameters \( a \) and \( \kappa \) are as follows: \( a = 2.3 \) m and \( \kappa = 2.2 \) in Figs 3(a, b, c, e), \( a = 2.9 \) m and \( \kappa = 2.2 \) in Fig. 3(d), and \( a = 2.4 \) m and \( \kappa = 2.4 \) in Fig. 3(f). The same ellipse:

\[
[R_p(\theta), Z_p(\theta)] = [6.0 + 1.0 \cos \theta m, 2.0 \sin \theta m]
\]

in cylindrical co-ordinates, was used for \( \partial \Omega_p \) in Figs 3(a, b, d, e), and a circle with a 1.5 m radius was used in Fig. 3(c). The number of discretized points independent of each other on \( \partial \Omega_p \) was 16. These were located at the same interval with respect to the parameter \( \theta \). The influence of the change of either \( \partial_p \) or \( \partial \Omega_b \) was found to be very small. The influence of the sensor–plasma distance without noise is used to test the influence of the line density of the sensors on the identification accuracy. Comparison of Figs 3(d) and 3(a) shows that the influence is small. In contrast, the influence of the sensor–plasma distance with \( \pm 2\% \) noise was found to be very strong. By comparing Figs 3(d) and 3(e), a 20 cm error at the maximum in Fig. 3(d) is observed. The comparison with Fig. 3(e) shows that this method is robust against \( \pm 1-2\% \) noise in the case where the sensors are located close to the plasma.

Figure 3(f) shows the reproduction of a circular plasma. Twenty-four \( B_i \) sensors were located asymmetrically to the midplane along \( \partial \Omega_b \) at the same interval with respect to the parameter \( \theta \). A circle with a radius of 1.0 m was used for \( \partial \Omega_p \). The number of independent discretized points on \( \partial \Omega_p \) was eight; these were located at the same interval in polar co-ordinates with respect to the angle \( \theta \).

Fifty \( B_i \) sensors were necessary for accurate shape reproduction in JT-60U. The accuracy of the shape reproduction in ITER (48 \( B_i \) sensors) is illustrated in Fig. 3. If the identification errors must be decreased or if the environment has strong noise and/or unknown eddy currents, then an appropriately large number of sensors are needed. The sensor locations should also be properly determined. At this point, the BIE method can be a useful tool for obtaining the logical determination of the sensor locations, according to the following logic.

The identification errors in an error-free environment are produced mainly by the interpolation process (B-spline interpolation is adopted for a periodic function [14]). Therefore, if the flux intensity between a pair of adjacent \( B_i \) sensors is precisely reproduced using a given interpolation function, the error is minimized. On the other hand, if the flux intensity is not precisely reproduced, one or more additional sensors are needed between the adjacent sensors. This is in agreement with the natural conviction that many sensors are needed in an area having strong non-linearity. Furthermore, in an actual system, errors due to noise and mechanical tolerances (coil and sensor placement and orientation) must be taken into account to a larger extent than the error arising from the interpolation process. Consequently, more sensors may be provided at the proper locations, as needed for the required accuracy.

The BIE method gives an accurate identification for ITER. While the LFE method includes a large identification error, as mentioned in Section 1, it gives good results for applications to JT-60U. We should ask what the real cause of the accuracy deterioration is. The eigenfunctions in the LFE method are used to reproduce the flux field in a vacuum. Consequently, they determine both the flux distribution and the flux intensity distribution simultaneously with the determination of one unknown coefficient value. Such limitations should be compensated by increasing the number of the eigenfunctions. However, the LFE method has difficulties in the numerical computation of higher modes of eigenfunctions. These difficulties seem to reduce the accuracy of the identification of an ITER plasma with \( \kappa = 2.2 \). Since the portion of higher modes in JT-60U is small (\( \kappa = 1.5 \)), several lower modes of the eigenfunctions in LFE are sufficient to reproduce the vacuum magnetic field.

In contrast, in the BIE method, the flux and the flux intensity on the inner boundary \( \partial \Omega_p \) are independent variables in the boundary integral equations. They are determined so that the identified vacuum magnetic field agrees with the magnetic measurements. Therefore, this method has a wider range of freedom than the LFE method. It should be understood that this is the reason why the BIE method can identify an ITER plasma much more accurately than the LFE method when the same number of sensors are used.

5. CONCLUSIONS

On the basis of our studies, as discussed in the previous sections, we have reached the following conclusions.
A necessary condition for identification of the shape of the plasma surface is that either the flux intensities tangential to the curved surface $\partial \Omega_s$ or the flux function values at points on $\partial \Omega_s$ are continuously given, and that the current density distribution in the region is completely known, except for the plasma. The sensor surface $\partial \Omega_s$ is the closed curved surface along which the magnetic sensors are located. Measurement of the plasma internal quantities is unnecessary for shape identification.

The BIE method, which uses integral equations for the derivation of the defined necessary condition, is proposed for tokamak plasma shape identification. Application to JT-60U ($\kappa \approx 1.5$) and ITER ($\kappa \approx 2.2$) shows that the BIE method identifies the plasma shape more accurately, even with a finite number of magnetic sensors along $\partial \Omega_s$. If this method is used to numerically compute the exact analytical solution of the partial differential equation concerned and if the proper number of sensors are properly located along $\partial \Omega_s$, the plasma shape is definitely and accurately reproduced independently of the size or shape of the tokamak. Furthermore, several test calculations with the BIE method indicate the following features:

(a) The differences in the plasma current profiles have little influence on the accuracy of the shape identification.
(b) The influence of the change in the poloidal beta on the plasma shape identification is very small.
(c) The influence of the change in the hypothetical plasma surface ($\partial \Omega_s$) figure on the plasma shape identification is very small.
(d) The influence of the sensor–plasma distance without noise on the plasma shape identification is small. The influence of the line density of sensors in an ideal error-free environment can also be small. In contrast, the influence of the sensor–plasma distance with $\pm 2\%$ noise was found to be very strong.
(e) This method is robust against $\pm 1$–$2 \%$ noise in the case where the sensors are located close to the plasma.
(f) With this method, a small circular plasma can be identified by using a hypothetical plasma surface enclosed within the plasma.

Since in the BIE method the line integrals are numerically calculated using measured data, an appropriate number of sensors are needed for interpolation. The choice of the interpolation functions seems to include an uncertainty. However, this method can be a promising tool for estimating the number and locations of sensors necessary for the required accuracy of the plasma shape identification.

Appendix 1

SOLUTIONS OF A PDE
FOR PLASMA SHAPE IDENTIFICATION

From a mathematical point of view, the problem of identifying the outermost magnetic surface of the plasma is a type of boundary value problem of an elliptic second-order PDE. In general, there are three well known methods of solving a PDE:

(a) The method of separation of variables;
(b) The variational method → finite element method (numerical computation);
(c) The method of boundary integral equations (Green function method) → boundary element method (numerical computation).

All these methods originate from analytical and exact solutions of a PDE. In reality, the solution is numerically computed in either a finite series or at discretized points. The features of these methods and their applicability to shape identification are discussed below.

The method of separation of variables is based on the fact that variables in the Laplace operator ($\Delta$) are separable in Cartesian co-ordinates. Therefore, any set of variables resulting from conformal mapping of Cartesian co-ordinates are separable for the Laplacian. A PDE is converted to multiple ordinary differential equations (ODEs) corresponding to the number of variables in the co-ordinates being conformal to the Cartesian co-ordinates. The concerned operator in Eq. (2.6), i.e. $\text{rot rot} = \text{grad div} - \Delta$, is separable, though it is not identical with the Laplacian. The basis functions ('eigenfunctions') for Eq. (2.6) in Cartesian, cylindrical, toroidal and spherical co-ordinates are known. The solution of Eq. (2.6) is then composed of an infinite series of the eigenfunctions, whose linear coefficients are determined by the boundary condition. It is, however, impossible to determine an infinite number of coefficients from the boundary condition values, which are usually given at discretized finite points. Therefore, a large error may be included, especially in numerical calculations of higher modes of eigenfunctions [3]. This implies that unless the first several series of eigenfunctions can approximate the solution sufficiently accurately, the calculation will contain a fairly large margin of error. In fact, the LFE method identifies the shapes of low $\kappa$ ($< 1.8$) plasma very well. However, a plasma with high $\kappa$ ($\approx 2.2$) in ITER is inaccurately reproduced [9]. Thus, the method of separation of variables is not easily applicable to shape identification of highly elongated plasmas.
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The finite element method applied to the whole region including the plasma is known for current profile identification. However, this premises a fixed outermost plasma surface. Hence the method of weighted residuals is applied to Eq. (2.3) for a vacuum region surrounding the plasma. The current density is $\mathbf{j} = 0$, and then

$$\text{rot rot } \mathbf{A} = 0$$  \hspace{1cm} (A1.1)

The vector potential $\mathbf{A}$ is adopted for a weight function. After the two sides of Eq. (A1.1) are multiplied by $\mathbf{A}$, the volume integral of the two sides over the region $\Omega_{s-p}$ yields

$$\int_{\Omega_{s-p}} \mathbf{A} \cdot \text{rot} \text{ rot } \mathbf{A} \, dV = \int_{\Omega_{s-p}} (\text{rot } \mathbf{A})^2 \, dV \qquad (A1.2)$$

where $\partial \Omega_s$ and $\partial \Omega_p$ are the sensor surface and the hypothetical plasma surface, respectively, and $\Omega_{s-p}$ is the region bounded by both $\partial \Omega_s$ and $\partial \Omega_p$, as shown in Fig. 1. Equation (A1.2) is discretized on the finite elements over the region $\Omega_{s-p}$ and is solved numerically. The boundary value $\mathbf{A}$ on the surfaces $\partial \Omega_s$ and $\partial \Omega_p$ is required for solving Eq. (A1.2). The value of $\mathbf{A}$ on $\partial \Omega_p$, however, cannot be fixed beforehand. Consequently, this approach encounters difficulties when it is applied to an unfixed boundary value problem.

The method of boundary integral equations is based on an exact formula of the solution of a PDE using a Green's function with both Dirichlet and Neumann boundary conditions. Part of the boundary shape and value cannot be fixed (this situation is the same as that mentioned above). However, the following features exist in the method of boundary integral equations (1). The Dirichlet and Neumann conditions on the surface $\partial \Omega_s$ can be derived from each other by putting the boundary $\partial \Omega_s$ (enclosing the analytical region) analytically and infinitely far from the concerned region (see Fig. 1). (2) The Dirichlet and Neumann conditions on $\partial \Omega_p$ can be computed using these conditions on $\partial \Omega_s$. The first feature is an ‘infinitely far boundary’, which is one of the difficulties of numerical computation. The second feature implies that flux function extrapolation towards the plasma is possible. Furthermore, no difficulty arises in the numerical computation of the solution expressed as a complete formula because a boundary integral with singularity is stable. Since the hypothetical plasma boundary $\partial \Omega_p$ must be located sufficiently inside a plasma, it is necessary to roughly know the plasma shape before identification.

Appendix 2

INFINITE SERIES FORMULA OF THE LINE INTEGRAL ALONG THE INTERVAL INCLUDING A SINGULAR POINT

It is well known that a definite integral of a continuous regular function can be expressed as an infinite series formula. We discuss here the question of whether the result of the integral can be expressed as an infinite series formula when the integrand with a singular point diverges to infinity and when the definite integral is bounded in the sense of the Riemannian integral.

Singular integrals introduced for boundary integral equations are

$$S_{\phi \gamma} = \int_{\partial \Omega} G(\mathbf{x}, \mathbf{y}) \cdot \text{grad } \phi(\mathbf{y}) \cdot \frac{d\mathbf{S}(\mathbf{y})}{r_1^2} \qquad (A2.1)$$

$$S_{\phi \gamma} = \int_{\partial \Omega} \phi(\mathbf{y}) \cdot \text{grad } G(\mathbf{x}, \mathbf{y}) \cdot \frac{d\mathbf{S}(\mathbf{y})}{r_1^2} \qquad (A2.2)$$

where $\partial \Omega_s$ is a closed surface including a singular point on the poloidal cross-section. Now $\partial \Omega_s$ is divided into two regions: a sufficiently small region involving a singular point ($\partial \Omega_s$) where the flux function is presumed to be constant, and the remaining region ($\partial \Omega_w$), i.e. $\partial \Omega_s = \partial \Omega_s + \partial \Omega_w$. The integral over $\partial \Omega_w$ is a normal integral and can be expressed as an infinite series. The closed surface $\partial \Omega_s$ is approximated by the polygon in the same manner as in Section 2 and is shown in Fig. 4. The singular point $\bar{x}$ is presumed to be the middle point of the side $P_N P_1$, whose length

FIG. 4. N-sided polygonal approximation of the closed surface and the definitions of points and a vector.

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is $2\pi h$. The side $P_NP_i$ is taken as $\partial \Omega_v$. This region is sufficiently small that $S_{Gv}$ and $S_{qvG}$ are approximated as

$$\int_{\partial \Omega_v} G(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \cdot \mathbf{n} \cdot \phi(\bar{\mathbf{y}}) \cdot \frac{dS(\bar{\mathbf{y}})}{r_y} \approx \text{grad} \phi(\bar{\mathbf{x}})$$

$$\times \int_{-h}^{h} G(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \cdot \mathbf{n} \cdot \frac{2\pi ds}{r_y} \tag{A2.3}$$

where the variable $s$ in the integrands is the distance along the side $P_NP_i$ from the middle point $\bar{\mathbf{x}}$ whose position vector is $\bar{\mathbf{x}} = (r_x, z_x)$, $\bar{\mathbf{y}} = (r_y, z_y)$, $r_y = r_x + s \cos \eta$, $z_y = z_x + s \sin \eta$, $\mathbf{n}$ is a vector normal to the side $P_NP_i$ in the direction towards the plasma, as shown in Fig. 4, and the slope of the side is $\tan \eta$. The right hand sides of Eqs (A2.3) and (A2.4) are rewritten as follows:

$$\int_{-h}^{h} G(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \cdot \mathbf{n} \cdot \frac{2\pi ds}{r_y} = 8\pi \left( \frac{\sin \eta}{-\cos \eta} \right)$$

$$\times \int_{0}^{h} K(k) ds - 16\pi \left( \frac{\sin \eta}{-\cos \eta} \right) \int_{0}^{h} E(k) ds \tag{A2.5}$$

$$\int_{-h}^{h} \text{grad} G(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \cdot \mathbf{n} \cdot \frac{2\pi ds}{r_y} = 4\pi \sin \eta$$

$$\times \int_{0}^{h} K(k) ds - \frac{8\pi \sin \eta}{r_x} \int_{0}^{h} E(k) ds \tag{A2.6}$$

Using approximations of $k \approx 1 - (s^2/8r_x^2)$ and $k^2 \approx 1 - (s^2/4r_x^2)$, the complete elliptic integrals of the first and second kind, $K(k)$ and $E(k)$, are numerically integrated as

$$\int_{0}^{h} K(k) ds = \sum_{i=0}^{10} \left( \frac{1}{8r_x^2} \right)^i$$

$$\times \left[ p_{Ki} - q_{Ki} \left\{ \log \left( \frac{h^2}{8r_x^2} \right) - \frac{2}{2i + 1} \right\} \right] \frac{h^{2i+1}}{2i + 1} \tag{A2.7}$$

$$\int_{0}^{h} E(k) ds = \sum_{i=0}^{10} \left( \frac{1}{8r_x^2} \right)^i$$

$$\times \left[ p_{Ei} - q_{Ei} \left\{ \log \left( \frac{h^2}{8r_x^2} \right) - \frac{2}{2i + 1} \right\} \right] \frac{h^{2i+1}}{2i + 1} \tag{A2.8}$$

where $p_{Ki}$, $q_{Ki}$, $p_{Ei}$ and $q_{Ei}$ are constant coefficients that were prepared for calculations of $K(k)$ and $E(k)$ with double precision (i.e. eight bytes = one word) [15].

As a result of the above derivations, Eqs (A2.1) and (A2.2) are found to have the same formula at $\bar{\mathbf{x}} = \alpha$:

$$\int_{\partial \Omega_v} H(\alpha, \beta) q(\beta) \, d\beta = \int_{\partial \Omega_w} H(\alpha, \beta) q(\beta) \, d\beta \tag{A2.9}$$

$$+ \int_{\partial \Omega_v} H(\alpha, \beta) q(\beta) \, d\beta$$

The first and second terms in Eq. (A2.9) correspond to those in Eq. (A2.10). The bounded function $Y(h, \alpha)$ is easily derived from Eqs (A2.5) to (A2.8). By using the following definitions in the second term of Eq. (A2.10): $\beta_N = \alpha$, $h_N = h$ and $w(h_N, \alpha, \alpha) = Y(h, \alpha)$, the right hand side of Eq. (A2.10) results in

$$\sum_{i=1}^{N} w(h_i, \alpha, \beta_i) q(\beta_i) \tag{A2.10}$$

Increasing $N$ to infinity, Eq. (2.17) is obtained:

$$\int_{\partial \Omega_v} H(\alpha, \beta) q(\beta) \, d\beta = \sum_{i=1}^{\infty} w(h_i, \alpha, \beta_i) q(\beta_i) \tag{A2.11}$$

In an actual calculation, Eq. (A2.10) is used instead of Eq. (A2.11). The interval of the integration, $\partial \Omega_v$, should be determined to be sufficiently small that $\phi$ can be regarded as a constant in $\partial \Omega_v$, while the integrand must not overflow numerically in the interval $\partial \Omega_w$.

Appendix 3

RESULT OF THE BIE METHOD

The result of the BIE method gives an exact solution for a vacuum region with the hypothetical plasma surface located inside the plasma. Plasma shape identification based on boundary integral equations (the BIE method) requires the hypothetical plasma surface to be located inside the plasma. The definitions are repeated as follows: $\partial \Omega_p$ is the hypothetical plasma surface completely enclosed by the plasma, $\partial \Omega_P$ is the real plasma surface and $\partial \Omega_s$ is the surface along which the magnetic sensors are located. The solution of Eq. (2.4) including $\partial \Omega_P$ is defined as $\xi(\bar{\mathbf{x}})$ and can be expressed as

$$\sigma \xi(\bar{\mathbf{x}}) = \int_{\partial \Omega_p} [G \text{ grad } \xi - \xi \text{ grad } G] \cdot \frac{dS(\bar{\mathbf{y}})}{r_y}$$

$$+ \int_{\partial \Omega_{p-w}} \mu_0 (j_e + j_i) G \cdot \frac{dV(\bar{\mathbf{y}})}{r_y} \tag{A3.1}$$

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where the vector \( \vec{S} \) is normal to the surface in the direction towards the plasma, \( j_e \) and \( j_i \) are the known current densities in the regions \( \Omega_{p-e} \) and \( \Omega_{p-i} \), respectively, and \( \sigma = \{ 8\pi^2(\Omega_{p-e} \supset \bar{x}), 4\pi^2(\partial \Omega_p \supset \bar{x}), 0 \} \) (exterior to \( \Omega_{p-e} \)). On the contrary, by assuming that the plasma current density \( j_p \) is known, the solution \( \phi(\bar{x}) \) is given by

\[
\sigma \phi(\bar{x}) = \int_{\partial \Omega_p} \left[ G \frac{\nabla \phi - \phi \nabla G}{r_p^2} \right] \cdot \frac{\vec{S}(\bar{y})}{r_p} \, dS(\bar{y}) \\
+ \int_{\partial \Omega_p+} \mu_0(j_c + j_v) \cdot G \frac{dV(\bar{y})}{r} \\
+ \int_{\partial \Omega_p-} \mu_0(j_c + j_v) \cdot G \frac{dV(\bar{y})}{r} \quad (A3.2)
\]

By introducing the surface \( \partial \Omega_{p+} \), Eq. (A3.2) can be rewritten as

\[
\sigma \phi(\bar{x}) = \sigma^e \psi(\bar{x}) \\
+ \int_{\partial \Omega_{p+}} \left[ G \frac{\nabla \phi - \phi \nabla G}{r_p^2} \right] \cdot \frac{\vec{S}(\bar{y})}{r_p} \, dS(\bar{y}) \\
+ \int_{\partial \Omega_{p-}} \mu_0(j_c + j_v) \cdot G \frac{dV(\bar{y})}{r} \quad (A3.3)
\]

where \( \sigma^e \psi(\bar{x}) \) is defined as:

\[
\sigma^e \psi(\bar{x}) = \int_{\partial \Omega_p} \left[ G \frac{\nabla \phi - \phi \nabla G}{r_p^2} \right] \cdot \frac{\vec{S}(\bar{y})}{r_p} \, dS(\bar{y}) \\
- \int_{\partial \Omega_{p+}} \left[ G \frac{\nabla \phi - \phi \nabla G}{r_p^2} \right] \cdot \frac{\vec{S}(\bar{y})}{r_p} \, dS(\bar{y}) \\
+ \int_{\partial \Omega_{p-}} \mu_0(j_c + j_v) \cdot G \frac{dV(\bar{y})}{r} \\
- \int_{\partial \Omega_{p+}} \mu_0(j_c + j_v) \cdot G \frac{dV(\bar{y})}{r} \quad (A3.4)
\]

and \( \sigma^e = \{ 8\pi^2(\Omega_{p-e} \supset \bar{x}), 4\pi^2(\partial \Omega_p \supset \bar{x}), 0 \} \) (exterior to \( \Omega_{p-e} \)). In a vacuum region (exterior to \( \Omega_{p-e} \)), \( \sigma^e \psi(\bar{x}) = 0 \) in Eq. (A3.3). Consequently, Eq. (A3.3), in which the plasma current density is taken into account, is expressed for a vacuum region as

\[
\sigma \phi(\bar{x}) = \int_{\partial \Omega_p} \left[ G \frac{\nabla \phi - \phi \nabla G}{r_p^2} \right] \cdot \frac{\vec{S}(\bar{y})}{r_p} \, dS(\bar{y}) \\
+ \int_{\partial \Omega_{p-}} \mu_0(j_c + j_v) \cdot G \frac{dV(\bar{y})}{r} \quad (A3.5)
\]

On the other hand, by introducing the plasma surface \( \partial \Omega_{p+} \) in Eq. (A3.1), we obtain

\[
\sigma \xi(\bar{x}) = \int_{\partial \Omega_p} \left[ G \frac{\nabla \xi - \xi \nabla G}{r_p^2} \right] \cdot \frac{\vec{S}(\bar{y})}{r_p} \\
- \int_{\partial \Omega_{p+}} \left[ G \frac{\nabla \xi - \xi \nabla G}{r_p^2} \right] \cdot \frac{\vec{S}(\bar{y})}{r_p} \quad (A3.6)
\]

The summation of the first and second terms in Eq. (A3.6) is a non-zero value only inside the region \( \Omega_{p-e} \) and it vanishes in a vacuum, as also follows from a derivation using \( \sigma^e \psi(\bar{x}) \) in Eqs (A3.3) and (A3.4). Taking into account that there is no current flow in \( \Omega_{p-e} \), we obtain

\[
\sigma \xi(\bar{x}) = \int_{\partial \Omega_p} \left[ G \frac{\nabla \xi - \xi \nabla G}{r_p^2} \right] \cdot \frac{\vec{S}(\bar{y})}{r_p} \\
+ \int_{\partial \Omega_{p-}} \mu_0(j_c + j_v) \cdot G \frac{dV(\bar{y})}{r} \quad (A3.7)
\]

The right hand side of Eq. (A3.7) is identical with that of Eq. (A3.5). Consequently,

\[
\phi(\bar{x}) = \xi(\bar{x}) \quad (A3.8)
\]

Therefore, it is understood that with the hypothetical plasma surface located inside the plasma, the result from the BIE method gives an exact solution for the vacuum region.

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