A Method for the Identification of the Plasma Magnetic Boundary in Machines for Fusion Research

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Abstract—The paper deals with a method for the identification of the plasma magnetic contour in machines for fusion research using magnetic measurements. The method is based on the integration of an electromagnetic model of the passive structures surrounding the plasma and of an equivalent plasma representation able to reproduce the magnetic configuration measured at the probe locations. No physical equilibrium is considered for the plasma and therefore the results of the identification are meaningful outside the plasma contour only.

Index Terms—Inverse problems, magnetic fusion research, quasistatic magnetic fields.

I. INTRODUCTION

IN THE TOROIDAL machines for magnetic fusion research a number of plasma parameters need to be measured or calculated in real time with a good accuracy in order to control the discharge evolution and to optimize its performance. One of the most relevant items to identify is the shape of the plasma magnetic contour, defined as the closed flux surface fully enclosed in the vacuum vessel and containing the whole plasma current. The plasma magnetic boundary can be estimated starting from a number of external measurements provided by magnetic probes located around the plasma. A number of equilibrium codes, based on the force balance, have been developed for Tokamaks [1] and for RFP machines [2], to fit the magnetic measurements with the magnetic field given by a proper plasma model. Unfortunately an accurate reconstruction of the plasma equilibrium is time consuming and therefore is normally performed off-line.

To compute the plasma magnetic contour in a way fast enough to be used in a feedback control, simpler methods have to be implemented. Moreover the plasma is magnetically strongly coupled with the passive structures, such as the vacuum vessel or the stabilising shell, so that the eddy currents in these components can be significantly large during fast electromagnetic transients; therefore the simplified methods must also take into account their effects.

Several techniques have been suggested to identify the evolution of the plasma magnetic boundary from external magnetic probe signals in presence of eddy currents [3], [4]. The identification method presented here is based on an electromagnetic model of the passive structures and an equivalent plasma representation able to approximate the magnetic configuration of the plasma at the probe locations.

II. DESCRIPTION OF THE IDENTIFICATION PROBLEM

The plasma magnetic contour, $\gamma_p$, has to be identified, as a function of time, from $M$ external magnetic measurements, Fig. 1, under the following hypothesis:

i) axisymmetry of the induction field and of the current densities;
ii) linearity of the media;
iii) evolution of the plasma as a sequence of magnetohydrostatic configurations.

The input quantities of the inverse identification problem are the magnetic measurements of the poloidal induction field component $B_m(t)$ and of the poloidal flux $\varphi_m(t)$ in a number of observation points $P$ located around the plasma region; in the experimental machines a Rogowski coil is also present to measure the total plasma current. The magnetic measurements are arranged in the vector $\mathbf{u}_m$. The active toroidal currents $i_{a1}(t)$, which are also measured, are arranged in the vector $\mathbf{i}_a$. Usually, in the experimental machines for fusion research, the time derivatives of $\mathbf{i}_a$, $\mathbf{u}_m$ are measured also. The toroidal current density in the passive conductors $\mathbf{J}$ and in the plasma $\mathbf{J}_p$ are unknown quantities.

In general the solution of this inverse problem from a finite number of external magnetic measurements is not unique. To
obtain a unique plasma magnetic contour, approximating the actual plasma boundary, a regularization function has to be introduced; in the paper a regularized least squares solution method is adopted. Moreover, from external magnetic measurements only, the actual internal plasma current $J_p$ can not be reconstructed.

A. Electromagnetic Model of the Passive Conductors

A simplified model of the passive structures has been introduced to account for their effect. The current density in the passive conductors has been discretized in $N$ axisymmetric elementary flux tubes of quadrilateral cross-section and uniform current $i_k(t)$ grouped in the vector $\mathbf{i}(t)$ (elementary currents). The electromagnetic coupling of these currents with the active currents and with the plasma can be described by means of an equivalent electrical network with lumped parameters, Fig. 2.

The corresponding equations are:

$$ \mathbf{Ri} - \mathbf{e} = \mathbf{v} $$

(1)

where

- $\mathbf{v}$ is the vector grouping the voltages at the leads $A_k$, $B_k$ of each branch;
- $\mathbf{R}$ is the matrix of the resistances of each elementary flux tube;
- $\mathbf{e} = -d\varphi / dt$ is the vector of the e.m.f. Introducing the mutual inductance matrices between the passive conductors $\mathbf{M}$ and between the active and the passive conductors $\mathbf{M}_a$,

the total flux vector can be written as:

$$ \varphi(t) = \mathbf{M} \mathbf{i} + \mathbf{M}_a \mathbf{i}_a + \varphi_p(t) $$

(2)

where the vector $\varphi(t) = \varphi(t) + \varphi_a(t) + \varphi_p(t)$ represents the total induction flux linked with each elementary current $\mathbf{i}$ and produced respectively by all the elementary currents $\mathbf{i}_a$ by the active currents $\mathbf{i}_a$ and by the plasma current.

The topology of the equivalent electrical network depends on the topology of the passive structures; two kinds of conductor typings have been considered:

a) toroidally closed conductors, like the vacuum vessel or the conducting plates;

b) toroidally open conductors like the stabilizing shell.

In the case a) of closed structures, each branch of Fig. 2 is short-circuited and $\mathbf{v} = \mathbf{0}$ in (1). In the case b) of open conductors, the presence of the toroidal gap can be approximated with a couple of knots $A$, $B$; the voltage between $A$, $B$ is $\mathbf{v}$ while the net current vanishes: $\sum_k i_k = 0$.

The voltage $\mathbf{v}$ can be eliminated from (1) so that an homogeneous system is obtained again: $\mathbf{R} \mathbf{i} - \mathbf{e}^a = \mathbf{0}$, where $\mathbf{R}^a$ and $\mathbf{e}^a$ are linear combinations of the rows of $\mathbf{R}$ and $\mathbf{e}$ respectively; $\sum_k i_k = 0$ must be considered as an additional relation.

Both the developed equivalent electrical networks of the two typings a) and b) provide an approximated representation of the effect of passive conductors; in particular the representation of the open conductors is meaningful away from the toroidal gap only.

The lumped parameters model (1), (2) developed for open or closed passive conductors leads to the following dynamic system:

$$ \mathbf{M} \frac{d\mathbf{i}}{dt} + \mathbf{Ri} = -\mathbf{M}_a \frac{d\mathbf{i}_a}{dt} - \frac{d\varphi_p}{dt} $$

(3)

being $\mathbf{M}$, $\mathbf{R}$ the inductance and resistance square matrices of the passive conductors; $\varphi_p(t)$ is the vector of the fluxes linked by the passive conductors and due to the plasma current. The system (3) can be thought as a constraint to which the passive conductor currents must comply with.

The mutual inductance between a pair $i$, $j$ of axisymmetric conductors of quadrilateral cross-section with uniform current densities are numerically computed according to the formula:

$$ M_{ij} = \frac{1}{A_i A_j} \sum_{m,n} \sum_{h,k} w_i^m w_j^n w_i^h w_j^k \varphi \left( P_{m,n}^i; Q_{h,k}^j \right) $$

(4)
being $A_i$, $A_j$ the area of the cross-section of the conductors; 
\[ \varphi(P_i, n; \Omega^{\text{m,n}}_k) = 2\pi r_i^2 n\varphi_i(P_i, n), \quad \Omega^{\text{m,n}}_k \] is the flux linked by an azimuthal circumference of radius $r_{m,n}$ passing through $P_i, n$ and produced by a unit filamentary azimuthal current in $\Omega^{\text{m,n}}_k$. $Q^{\text{m,n}}_k$ are the Gauss integration points located in the cross-sections of the $i$, $j$ conductors respectively; the coefficients $w_i$ are the Gauss weights.

In the case of the self-inductance ($i = j$) the order of the two quadrature formulas in (4) is set different to avoid singularities.

**B. Representation of the Plasma Effect**

To represent the effect of the toroidal plasma current density $J_p$, a number $N_p$ of toroidal equivalent plasma currents is considered, located inside the vacuum vessel [5]. These currents $i_p(t)$ have to be determined in such a way that the total induction field configuration produced by the active, passive and equivalent currents best fits the magnetic measurements. The equivalent currents are a discrete basis used to approximate the magnetic configuration produced by the actual plasma current distribution outside $\gamma_p$. Therefore in order to obtain a good approximation of the magnetic contour, the selected basis must be able to represent the multipolar moments [6] of the current density $J_p$.

The contributions to the magnetic measurements $u_m$ of the active currents $i_a$, passive conductors currents $i$, and of the equivalent plasma currents $i_p$ can be expressed as:

\[ G_i + G_p\dot{i}_p = u_m - G_a\dot{i}_a \]  

where the matrices $G$, $G_p$, $G_a$ link each source current to the measurements $u_m$ of the poloidal flux and of the induction field component on each probe. The elements of the $G$ matrices are numerically computed, by means of Gaussian integration formulas, as in (4), which depend on the kind of the probe: in the case of a flux probe in $P$, the flux $\varphi$ linked by a circumference passing through $P$ and produced by an axi-symmetric coil of quadrilateral cross-section with uniform unit current density is computed; in the case of a pick-up coil in $P$ the induction component on it is computed.

The $G$ matrices are not square being in principle the number of measurements independent of the number of currents. In particular the number of equivalent currents $N_p$ is chosen to be not larger than the number of measurements $M$: therefore the matrix $G_p$ has $M \times N_p$ dimension.

**C. Integration of the Passive Conductor Model and of the Plasma Equivalent Representation**

As far as the passive conductor model is concerned, the flux $\varphi_p$ in (2), (3) can be written as:

\[ \varphi_p = M_p\dot{i}_p, \]  

being $M_p$ the mutual inductance matrix between the elementary currents $\dot{i}$ and the equivalent plasma currents $\dot{i}_p$. The vector of the elementary currents in the passive conductors is derived as:

\[ \dot{i} = -G^T G_p \dot{i}_p + G^T u_m - G^T G_a \dot{i}_a \]  

by computing, from (5), the Moore–Penrose pseudo inverse matrix $G^+$; substituting it in (3), the following linear dynamic system is obtained:

\[ A\frac{d\dot{i}}{dt} + B\dot{i} = T_1\dot{i}_a + T_2\dot{i}_a + T_3\dot{i}_m + T_4 u_m \]  

where: $A = (M_p - MG^+ G_p)$, $B = -RG^+ G_p$ are matrices of $N \times N_p$ dimension. The right-hand side of (7) is a known vector depending on the magnetic measurements and on the active currents and their time derivatives; the matrices are defined as:

\[ T_1 = RG^+, \quad T_2 = -MG^+, \quad T_3 = (MG^+ G_a - M), \quad T_4 = RG^+ G_a. \]

From (7) and from the given initial condition vector $\dot{i}_p(0)$, the vector $\dot{i}_p(t)$ of plasma equivalent currents can be derived by solving at each integration step an over-determined ($N > N_p$) least squares problem by means of the pseudo-inverse matrix $A^+$ of $A$: as numerical integration scheme a fourth order Runge–Kutta algorithm has been used.

As regularization strategy in the solution of the least squares approximations, the truncated pseudo-inverse are computed by forming the truncated Singular Value Decomposition [7] of the matrices:

\[ G = U S_h V^T, \quad A = P S_h Q^T \]

being $U, V, Q$ orthonormal square matrices, and $S_h, S_k$ the diagonal matrices of the truncated singular values; the truncation consists in setting to zero the singular values below $s_h, s_k$.

A stable and smooth solution is obtained, in our case, when $s_h, s_k$ are set around some percent (1–5 %) of the corresponding maximum singular value.

The currents in the passive conductors $i$ can be computed following two different approaches, depending on what it is needed:

1) *The magnetic contour only is required*

In this case (6) can be used to determine the vector $i$ from a best fitting solution; the vector $\dot{i}_p$ derives from the solution of (7). This way is very fast and the plasma magnetic boundary can be quickly determined from a flux map with an acceptable accuracy. However the computed passive currents often has no physical meaning being their spatial distribution affected by sharp oscillations.

2) *The magnetic contour and the passive currents are required*

In this case the model of the passive conductors (3) can be profitably used as physical constraint:

\[ M \frac{d\dot{i}}{dt} + R\dot{i} = -M\dot{i}_a + M_p \frac{d\dot{i}_p}{dt} \]

and the vector $\dot{i}$ can be determined by a further numerical integration of the above differential system. The simplified model of the passive conductors acts as a regularization function providing a smooth and physical solution for $i$. This second integration is much slower than the former used to solve (7) because $M$ can be a large square matrix according to the discretization of the passive conductors adopted. Again the magnetic contour can be deduced.
III. APPLICATION AND RESULTS

A numerical code based on the described method has been developed and tested against a number of reference magnetic configurations numerically generated for the ITER (International Thermonuclear Reactor) machine by means of the MHD equilibrium code MAXFEA, widely used in the engineering design phase of the machine. Starting from a flat-top condition of a 21 MA diverted plasma current (X-point configuration), a fault in the power supply control has been simulated. The plasma evolves shrinking its cross section and moving downward. The flat-top phase lasts 2 s while the simulation is stopped after 4.5 s. The simulated magnetic signals $u_m$ and $\frac{du_m}{dt}$ have been generated and used as input for the identification code together with the time evolution of the active current vectors $i_a$ and $\frac{di_a}{dt}$. At the time $t = 1$ s, the reference contour and two identified plasma contours are displayed in Fig. 5; they are computed according to the two different computation approaches 1) and 2) respectively. Between the reference and the reconstructed plasma contours, minor discrepancies (within a few cm at the worst location) are observed. The time evolution of the identified magnetic contour is shown, in Fig. 6, at three time instants: $t = 1$ s, $t = 3$ s and $t = 3.8$ s.

An important point in the solution of (7) is the determination of the initial condition vector $i_p(0)$; the initial currents in the equivalent conductors are not a priori known, but only their sum, equal to the total plasma current, is given. In the case considered here $i_p(0)$ has been approximated by solving the least squares problem (5) at $t = 0$:

$$i_p(0) = -G_p^T G i(0) + G_p^T u_m(0) - G_p^T G a i_a(0)$$

by computing the truncated pseudo-inverse of $G_p$. During the flat-top the currents $i(0)$ in the passive conductors are negligible and can be set to zero.

IV. CONCLUSIONS

The developed code provides results in a good agreement with the reference data. However further work needs to be done as concerns the regularization of the solution and the algorithm optimization in the perspective of the on-line identification.

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REFERENCES


